PLANE GEOMETRY

CHARLES SOLOMON
and
HERMAN H. WRIGHT

BOARD OF EDUCATION
CITY OF NEW YORK

*Public School No.*______ *Borough of*__________

NAME	CLASS	DATE ISSUED	CONDITION	
			WHEN ISSUED	WHEN RECEIVED
Foss Clifton	322			
Schamis, S.J.	413			

PLANE GEOMETRY

PLANE GEOMETRY

s 70403

BY

CHARLES SOLOMON
CHAIRMAN OF THE DEPARTMENT OF MATHEMATICS
BOYS HIGH SCHOOL, BROOKLYN, N. Y.

AND

HERMAN H. WRIGHT
DISTRICT SUPERINTENDENT ASSIGNED TO HIGH SCHOOLS
NEW YORK CITY

—PROPERTY OF—
BOARD OF EDUCATION
CITY OF NEW YORK

CHARLES SCRIBNER'S SONS
NEW YORK CHICAGO BOSTON ATLANTA
SAN FRANCISCO DALLAS

Copyright, 1929, by
CHARLES SCRIBNER'S SONS

Printed in the United States of America

PREFACE

The purpose of this book is to present geometry in such a way as to enable the student to do his own thinking and to apply his knowledge to practical problems. The usual text, after stating a proposition, proceeds to prove it by telling the student "Draw so and so; it then follows, etc." Why certain lines should be drawn, and why the procedure is as given, are profound mysteries to the average student. His function then becomes one of checking the author's procedure. This tends to make mere imitators of the students, and develops no power to handle original exercises. Indeed, the helplessness of the average student when confronted with an original exercise is well known to teachers of geometry. If, however, in the treatment of propositions, the student is trained to discover what lines must inevitably be drawn and what the method of procedure must be, he is at once put in the position of a discoverer, and original exercises may then be regarded as an intellectual treat.

The analytic method is the key to geometry, and it is the method here used. The analysis leads to a plan of procedure. Hence each proposition is preceded by an analysis, or plan, or both; and is followed by a synthetic proof. The synthetic proof is given as a model.

In this text the student is enabled to get his bearings immediately by a simple and concise statement of the nature of the subject-matter of geometry, why we study it, how we arrive at its truths, the need for a proof, what instruments are used, followed by twenty-five carefully chosen preliminary exercises. Thus the student is led to Proposition I without wasting many weeks.

Throughout this book the phraseology is adapted to the student of high school capability. The development of the text is psychological as well as logical. The proofs are simple, concise, and clear-cut. The exercises have been chosen and graded with much care, and are planned to develop the mathematical power of the student. Their number and variety afford ample practice for all types of courses. They range from the simple to those sufficiently difficult to test the capacity of the best students.

No work in geometry is really complete unless the student is required to apply his knowledge to practical problems. Hence at the end of each book is a group of real practical applications chosen from shop-practice, surveying, engineering, architecture, physics, and mechanics.

The selection and treatment of topics accord with the best modern usage. The subject of Loci is here treated fully, and, we believe, adequately. Construction problems, perhaps the most difficult part of geometry, are particularly stressed. This text is planned to meet the requirements of the average course and also of more exacting courses in elementary plane geometry.

The method used in this text is an outgrowth of many years of actual classroom teaching of geometry to thousands of boys and girls in our high schools. It has worked well. Therefore, we believe we have made a thoroughly teachable book which should lighten the teacher's labor and at the same time give the student a clear understanding of the principles of geometry.

C. S.
H. H. W.

CONTENTS

PAGE

INTRODUCTION 1–12

BOOK I — FIGURES FORMED BY STRAIGHT LINES

Fundamental Theorems 13–51
Fundamental Constructions 52–62
Additional Theorems 63–81
How to Attack Originals 82–84
Miscellaneous Exercises 85–89
Practical Applications 89–95

BOOK II — THE CIRCLE

Fundamental Theorems 96–136
Fundamental Constructions 137–144
How to Attack Construction Problems . . . 145–151
Problems of Construction 152–153
Loci 154–162
Miscellaneous Exercises 162–170
Practical Applications 171–174

BOOK III — PROPORTIONAL LINES AND SIMILAR POLYGONS

Fundamental Theorems and Constructions . . 175–211
Numerical Relations 212–219
Miscellaneous Exercises 220–232
Practical Applications 233–240

BOOK IV — AREAS OF POLYGONS

Fundamental Theorems and Constructions . . 241–260
Miscellaneous Exercises 260–270
Practical Applications 271–273

BOOK V — REGULAR POLYGONS AND THE CIRCLE

Fundamental Theorems and Constructions . . 274–297
Miscellaneous Exercises 297–305
Practical Applications 306–311

APPENDIX

PAGE
External Division — Harmonic Division . . . 312–314
Mean and Extreme Ratio — Application to Certain Regular Polygons 315–319
Perimeters of 2*n*-gons — Determination of value of π 320–323
Trigonometry of the Right Triangle 324–333
Important Formulas in Plane Geometry . . . 334–335
Table of Square Roots 336

INDEX 337

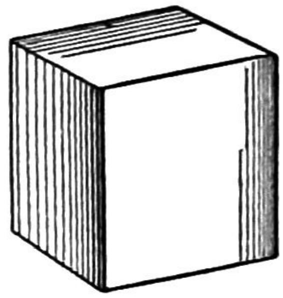
PLANE GEOMETRY

PLANE GEOMETRY

INTRODUCTION

1. The Subject-matter of Geometry. Geometry is the study which tells us about points, lines, surfaces, and solids (or any combination of these).

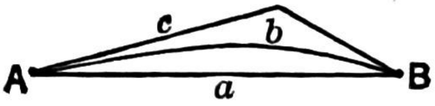
To get an idea of what we mean by these terms, imagine a block of wood in the shape of a cube. It is bounded by six flat surfaces called planes. The twelve edges are called straight lines. The eight corners are called points.

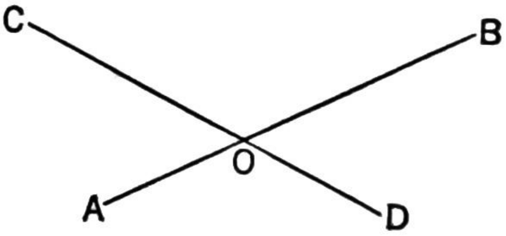


2. The Use of Geometry. Practically geometry is used by the surveyor, engineer, builder, technician, and others. But much more important than this is the training which it gives in careful, logical reasoning. This training in correct reasoning can be used in any field of human endeavor.

3. How We Arrive at the Truths of Geometry. In our study of geometry we begin with three facts which are taken for granted (called geometric axioms or postulates):

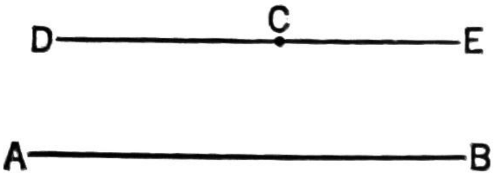
I. *Only one straight line can be drawn between any two points*, or we might say that two points determine a straight line. Thus, A and B are any two points. AB is the straight line joining them. Note that a straight line is the shortest distance between two points. In the annexed drawing, a represents a straight





line, *b* a curved line, *c* a broken line between points *A* and *B*. Note also that two straight lines can intersect in only one point. Thus *AB* and *CD* intersect at *O*.

II. *Through a given point only one straight line can be drawn parallel to a given straight line.* Thus, through *C* we can have only one straight line *DE* parallel to *AB*. (*Parallel lines* are lines in the same plane that can never meet, no matter how far they are prolonged.)



III. *Any geometric figure can be moved from one position to another without changing its size or shape.*

With these three truths as a basis, and with the aid of other truths, we arrive by careful, logical reasoning at many geometric facts. Some of the other truths here referred to are called general axioms, because they are used in all branches of mathematics.

4. General Axioms (used in all branches of mathematics).

Ax. 1. Things equal to the same thing, or to equal things, are equal to each other.

If $p = q$ and $r = q$, then $p = r$
or if $p = q$, $r = s$, and $q = s$, then $p = r$

Ax. 2. If equals are added to equals, the sums are equal.

If $a = b$ and $c = d$, then $a + c = b + d$

Ax. 3. If equals are subtracted from equals, the remainders are equal.

If $a = b$ and $c = d$, then $a - c = b - d$

Ax. 4. If equals are multiplied by equals, the products are equal.

If $a = b$ and $c = d$, then $a \times c = b \times d$
Hence it follows that like powers of equals are equal.
If $a = b$, then $a^2 = b^2$

Ax. 5. If equals are divided by equals, the quotients are equal.

If $a = b$ and $c = d$, then $\frac{a}{c} = \frac{b}{d}$

Ax. 6. The whole is equal to the sum of its parts, and is greater than any of its parts.

Ax. 7. A quantity may always be replaced by its equal. This is called substitution.

If $a + b = x$, and if $b = 5$, then $a + 5 = x$

Ax. 8. If one of three quantities is greater than the second, and the second is greater than the third, then the first is greater than the third.

If $a > b$, and $b > c$, then $a > c$

Ax. 9. If equals are added to unequals, the sums are unequal in the same order.

If $a > b$, and $c = d$, then $a + c > b + d$

Ax. 10. If equals are subtracted from unequals, the remainders are unequal in the same order.

If $a > b$, and $c = d$, then $a - c > b - d$

Ax. 11. If unequals are subtracted from equals, the remainders are unequal in the reverse order.

If $a = b$, and $c > d$, then $a - c < b - d$

Ax. 12. If unequals are added to unequals in the same order, the sums are unequal in that order.

If $a > b$, and $c > d$, then $a + c > b + d$

Ax. 13. If unequals are multiplied or divided by equals, the results are unequal in the same order.

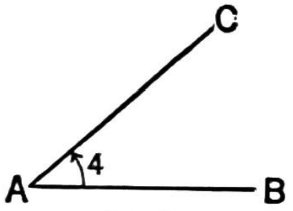
If $a > b$, and $c = d$, then $a \times c > b \times d$; also $\frac{a}{c} > \frac{b}{d}$

5. The Need for Proof. It is necessary to prove the truths of geometry because we cannot always trust our eyes, and because we cannot rely absolutely upon instruments of measurement. In fact no measuring instruments can give perfect results. But we can always rely upon the results obtained by correct reasoning.

6. Instruments Used. To make the drawings (constructions) of geometry, only two instruments are permitted to be used — the ruler or straightedge and compasses.

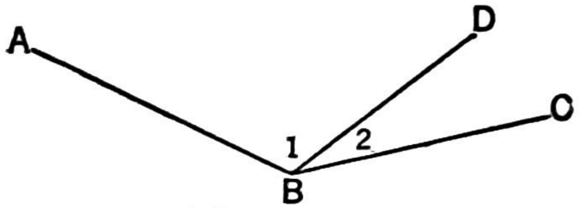
7. Fundamental Definitions. Certain terms are used throughout our work in geometry, and it is necessary for the pupil to have a clear idea of their meaning.

8. The two straight lines AB and AC meeting at A form *angle BAC*. AB and AC are called the *sides* of the angle, and point A is called the *vertex* of the angle. The angle is the amount of rotation that is necessary for AB to get into the position of AC (e.g. imagine the hands of a clock). Therefore the size of an angle is not affected when its sides are either lengthened or shortened. Very often it is convenient to use a number to designate an angle. Thus instead of saying "angle BAC," we might say "angle 4."



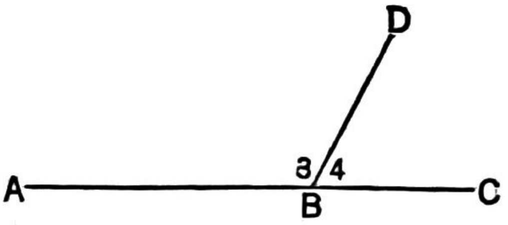
Angle

9. Angles 1 and 2 are called *adjacent angles* because they have a common side BD and a common vertex B.



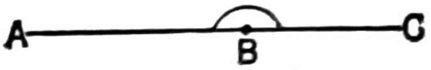
Adjacent Angles

10. Angles 3 and 4 are called *supplementary adjacent angles* because the sides BA and BC are in the same straight line and opposite each other.



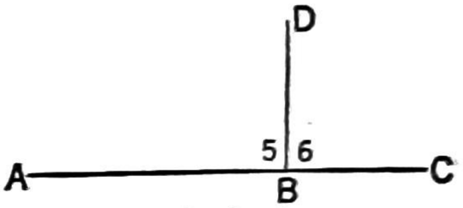
Supplementary Adjacent Angles

11. Angle *ABC* is called a *straight angle* because its sides *BA* and *BC* are in the same straight line and opposite.



Straight Angle

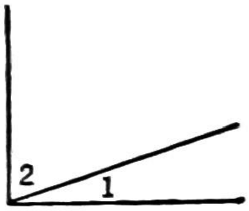
12. When two supplementary adjacent angles are equal, each is called a *right angle.* Thus angle 5 is a right angle; so is angle 6. A right angle is therefore half of a straight angle.



Right Angles

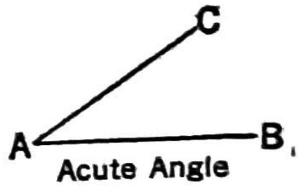
13. The sides of a right angle are said to be *perpendicular* to each other. Thus in the last diagram, *BC* is perpendicular to *BD*, and *BD* is perpendicular to *BC*.

14. Angles 1 and 2 are called *complementary angles* because their sum is a right angle.



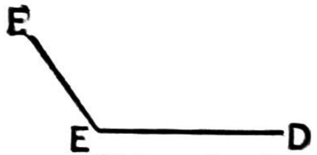
Complementary Angles

15. Angle *BAC* is called an *acute angle* because it is less than a right angle.



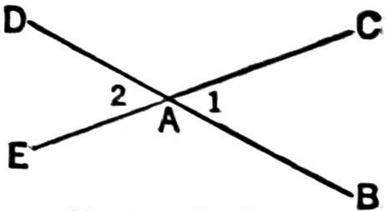
Acute Angle

16. Angle *DEF* is called an *obtuse angle* because it is greater than a right angle but less than a straight angle.

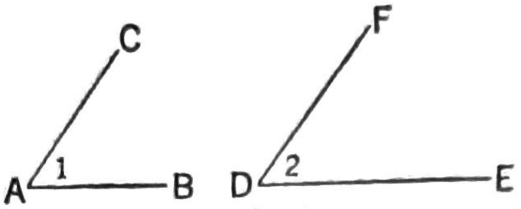


Obtuse Angle

17. If the sides of angle 1 are prolonged through the vertex *A* a pair of *vertical angles* is formed (angle 1 and angle 2). Can you mention another pair of vertical angles in this figure?



Vertical Angles

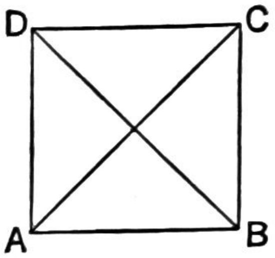


Equal Angles

18. If angle 1 can be placed upon angle 2 so that point A falls upon point D, AB along DE, and AC along DF, then we say angle 1 *equals* angle 2.

19. An angle is measured by finding how many times it contains the unit angle. The unit angle is $\frac{1}{90}$ of a right angle, and is called a *degree*. A degree is divided into sixty equal parts called *minutes*. Each minute is divided into sixty equal parts called *seconds*. An angle of 15 degrees, 12 minutes, and 42 seconds is written $15° 12' 42''$.

20. A geometric statement that requires proof is called a *theorem*. Every theorem consists of two parts: that which is *given*, and that which has *to be proved*. For instance, consider the following theorem: The diagonals of a square are equal. Here we are given the diagonals of a square, and we are asked to prove that they are equal. Using the annexed drawing, we would say



Given the square $ABCD$ with its diagonals AC and BD.

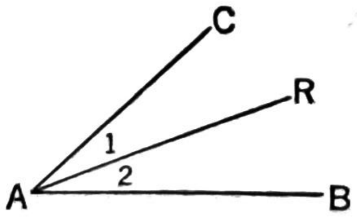
Prove $AC = BD$.

21. A geometric construction that is required to be made is called a *problem*. E.g. Draw a perpendicular to a given line from a given external point.

22. A *proposition* is either a theorem or a problem.

23. Any quantity is equal to itself. Thus we say $AB = AB$ *by identity*.

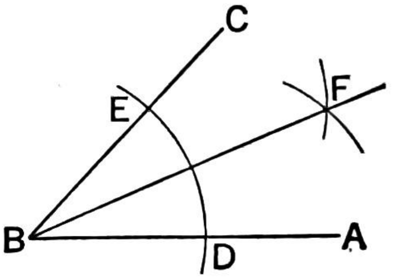
24. When a geometric figure is cut into two equal parts it is said to be *bisected*. Thus if angle 1 = angle 2, then angle BAC is bisected by the line AR. AR is called the *bisector* of angle BAC.



25. Simple Exercises in Geometric Drawing. The following simple exercises are given so that the pupil can become familiar with the use of the drawing instruments. For the present, the constructions are given without proof.

Problem 1. To bisect any angle, such as angle *ABC*.

Method: (*a*) With *B* as center, and any radius, draw arc *DE* cutting *AB* in *D* and *BC* in *E*.

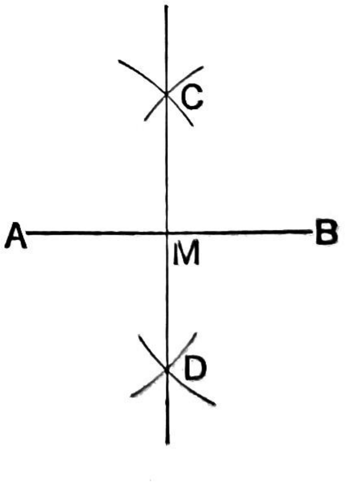


(*b*) With *D* as center, and a radius greater than half the distance *DE*, draw an arc.

(*c*) With *E* as center, and the radius used in (*b*), draw an arc cutting the last arc at *F*.

(*d*) Draw the straight line connecting *B* and *F*. Then *BF* is the bisector of angle *ABC*.

Problem 2. To bisect a given straight line *AB*.



Method: (*a*) With *A* as center, and a radius greater than one half *AB*, draw arcs above and below *AB*.

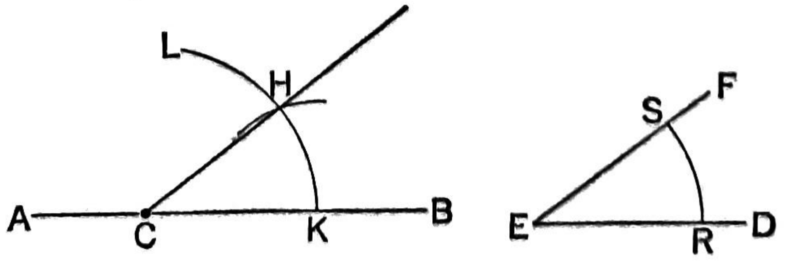
(*b*) With *B* as center, and the same radius, draw arcs cutting the other two at *C* and *D*.

(*c*) Draw the straight line connecting *C* and *D*, and let it cross *AB* at *M*.

Then *M* is the midpoint of *AB*.

Problem 3. From a point *C* in a given line *AB* to draw a line so as to make an angle with *AB* equal to angle *E*.

Method:



(*a*) With *E* as center, and any radius, draw arc *RS*.

(*b*) With C as center, and the same radius, draw arc KL.

(*c*) With K as center, and the distance RS as radius, draw an arc, cutting arc KL at H.

(*d*) Draw the straight line connecting C and H. Then angle BCH equals angle E.

Preliminary Exercises

(The word "line" means straight line, unless otherwise noted.)

Ex. 1. Draw two lines of unequal length, and then draw a line equal to their sum.

Ex. 2. Draw two lines of unequal length, and then draw a line equal to their difference.

Ex. 3. Draw two intersecting lines. Mention the pairs of adjacent angles, the pairs of vertical angles, the acute angles, the obtuse angles. When will each of the angles be a right angle?

Ex. 4. Draw an obtuse angle, and then draw its supplement. Is this supplement acute, right, or obtuse?

Ex. 5. Draw an acute angle, and then draw its supplement. Is this suplement acute, right, or obtuse?

Ex. 6. Draw an acute angle, and then draw its complement. Is the complement acute, right, or obtuse?

Ex. 7. Draw a given line, and divide it into four equal parts.

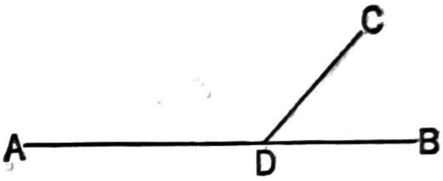
Ex. 8. Draw an obtuse angle, and divide it into four equal parts.

Ex. 9. Construct one half the supplement of a given acute angle.

Ex. 10. Draw two unequal angles, and then draw an angle equal to their sum; also an angle equal to their difference.

Ex. 11. Through how many degrees does the minute-hand of a watch move in 5 minutes? in 15 minutes? in 30 minutes? in 55 minutes?

Ex. 12. Two straight lines intersect, and one of the angles thus formed is 18°. Find the number of degrees in each of the other angles.



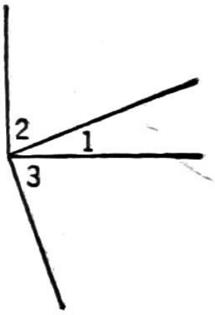
Ex. 13. CD meets AB so that angle ADC is three times angle BDC. How many degrees are there in each angle? 45°

Ex. 14. Express the complement of $x°$; the supplement of $x°$; the complement of $x°$ equals $\frac{1}{4}$ the supplement of $x°$. Solve this equation for x. 60°

Ex. 15. Prove that complements of the same angle are equal.

Given $\begin{cases}\text{angle 2 is the complement of angle 1.}\\ \text{angle 3 is the complement of angle 1.}\end{cases}$

Prove angle 2 = angle 3.



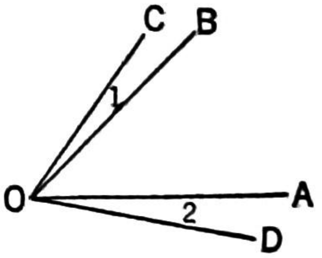
Proof.

(1) angle 2 = 90° − angle 1	(1) Given.
(2) angle 3 = 90° − angle 1	(2) Given.
(3) ∴ angle 2 = angle 3	(3) Axiom 1.

which was to be proved.

26. Note that a proof consists of a number of statements, and that each statement is supported by a reason. The reason must always be either something given, or something already proved, or a definition, or an axiom.

Ex. 16. In the annexed diagram angle 1 = angle 2. Prove that angle AOC = angle BOD.

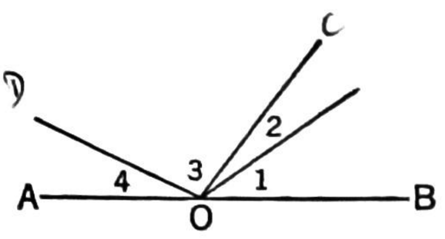


Ex. 17. Prove that complements of equal angles are equal.

Ex. 18. Prove that supplements of the same angle are equal.

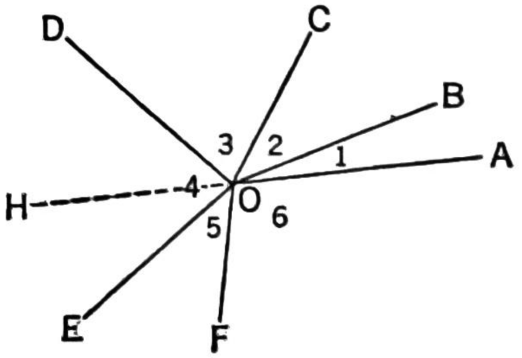
Ex. 19. Prove that supplements of equal angles are equal.

Ex. 20. Prove that the sum of all the angles about a point on one side of a straight line passing through the point is a straight angle or 180°.



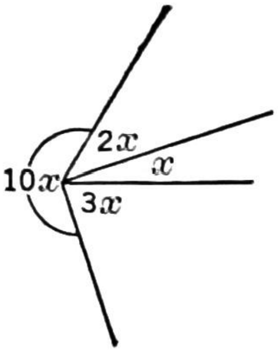
Ex. 21. Prove that the sum of all the angles about a point is two straight angles or 360°.

Sug. Prolong *AO* to *H*.

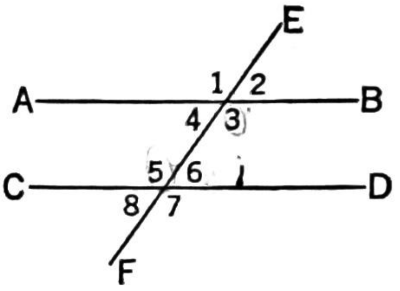


Ex. 22. There are twenty equal angles about a point in a plane. How many degrees are there in each angle?

Ex. 23. In the annexed diagram how many degrees are there in each of the four angles?



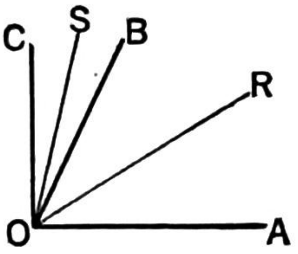
Ex. 24. *AB* and *CD* are two lines cut by a third line *EF*. If angle 3 = angle 5, prove angle 4 = angle 6 and that angle 2 = angle 8.



Ex. 25. Prove that the bisectors of two complementary adjacent angles form an angle of 45°.

Given angles *AOB* and *BOC* are complementary adjacent; *OR* bisects angle *AOB*; *OS* bisects angle *BOC*.

Prove angle *ROS* = 45°.



Proof.

(1) angle AOB + angle $BOC = 90°$	(1) Given.
(2) $\therefore \frac{1}{2}$ angle $AOB + \frac{1}{2}$ angle $BOC = 45°$	(2) Axiom 5.
(3) $\therefore$ angle ROB + angle $BOS = 45°$	(3) Axiom 7.
(4) $\therefore$ angle $ROS = 45°$	(4) Axiom 6.

which was to be proved.

Miscellaneous Exercises

(*a*) What is the complement of each of the following angles: 60°, 36°, 63°, 45°? What is the supplement of each of these angles?

(*b*) What angles do the hands of a clock make with each other when it is 2 o'clock, 4 o'clock, 6 o'clock, 9 o'clock?

(*c*) One of two supplementary adjacent angles is eight times the other. How many degrees are there in each angle? Illustrate your answer with a diagram.

(*d*) The supplement of an angle is five times its complement. Find the angle.

(*e*) Why must the sum of any two sides of a triangle be always greater than the third side?

(*f*) Points A, B, C, D, taken in order, are situated in a straight line. If $AB = CD$, prove that $AC = BD$.

(*g*) If the sides of an angle are lengthened or shortened, what effect is produced upon the angle?

SYMBOLS AND ABBREVIATIONS

27. To simplify the work, and to save space and time, the following symbols and abbreviations are used in this book:

+, −, ×, ÷, =, have the same meaning as in arithmetic.

>, is greater than
<, is less than
∴, therefore
∥, parallel, or is parallel to
∥s, parallels
⊥, perpendicular, or is perpendicular to
⊥s, perpendiculars
∠, angle ∠s, angles
△, triangle △s, triangles
▱, parallelogram
⊙, circle ⊙s, circles
⌒, arc
∼, similar, is similar to
≅, congruent, is congruent to
A′, A prime
cor., corollary
ax., axiom
prop., proposition
th., theorem
sug., suggestion
quad., quadrilateral
adj., adjacent
sup., supplementary, supplement

alt., alternate
int., interior
ext., exterior
corr., corresponding
Ex., exercise, example
def., definition
const., construction
fig., figure
hy., hypotenuse
iden., identity
isos., isosceles
rt. right
st., straight
sub., substitution
Q.E.D. which was to be proved (quod erat demonstrandum)
a. s. a., two angles and the included side of a triangle
s. a. s., two sides and the included angle of a triangle
s. s. s., three sides of a triangle

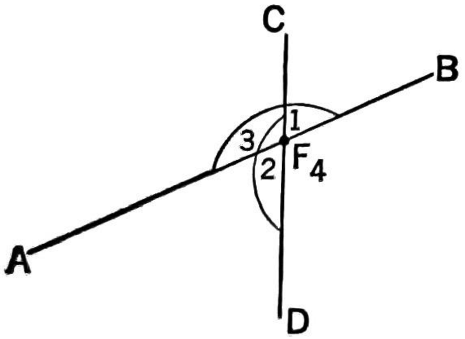
BOOK I

FIGURES FORMED BY STRAIGHT LINES

VERTICAL ANGLES

Proposition I. Theorem

28. If two straight lines intersect, the vertical angles are equal.



Given AB and CD, two straight lines that intersect at F.

Prove $\angle 1 = \angle 2$, $\angle 3 = \angle 4$.

Plan is to prove $\angle 1$ *and* $\angle 2$ *the supplement of the same angle.*

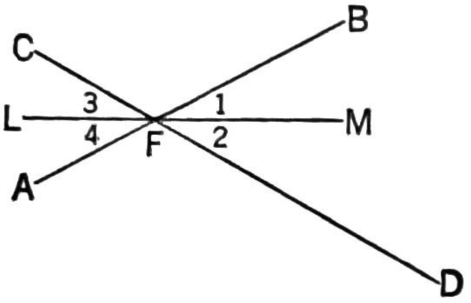
Proof.

(1) $\angle 1 + \angle 3 = \angle AFB$	(1) Ax. 6.
(2) $\angle 2 + \angle 3 = \angle DFC$	(2) Ax. 6.
(3) but $\angle AFB = \angle DFC$	(3) All straight angles are equal.
(4) $\therefore \angle 1 + \angle 3 = \angle 2 + \angle 3$	(4) Ax. 1.
(5) $\therefore \angle 1 = \angle 2$	(5) Ax. 3.

In like manner it can be proved that $\angle 3 = \angle 4$. (Let the pupil write out the proof for this.)

29. *Note:* Once a theorem has been proved, we can always make use of it in any example without going through the proof again.

Ex. 26. If a straight line bisects one of two vertical angles, it bisects the other also.



Given $\measuredangle$ BFD and CFA a pair of vertical angles, and that LM bisects $\angle BFD$.

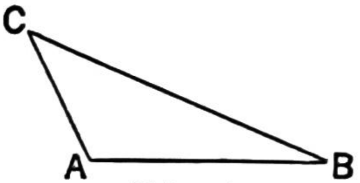
Prove LM bisects $\angle CFA$ (i.e. prove $\angle 3 = \angle 4$).

Proof.

(1) $\angle 3 = \angle 2$	(1) If two straight lines intersect, the vertical $\measuredangle$ are equal.
(2) $\angle 4 = \angle 1$	(2) Same reason.
(3) but $\angle 1 = \angle 2$	(3) Given.
(4) $\therefore \angle 3 = \angle 4$	(4) Things that are equal to equal things, are equal to each other.

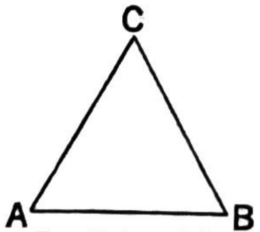
Q.E.D.

Definitions



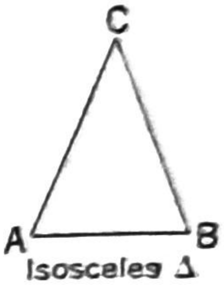
Triangle

30. ABC is called a *triangle.* Note that it is a closed figure bounded by three straight lines. AB, BC, AC are called the *sides* of the triangle. Points A, B, C are called the *vertices* of the triangle. $\angle A$, $\angle B$, $\angle C$ are called the *angles* of the triangle. Any side of the triangle may be considered as the base. The *base* is the side on which the triangle stands.



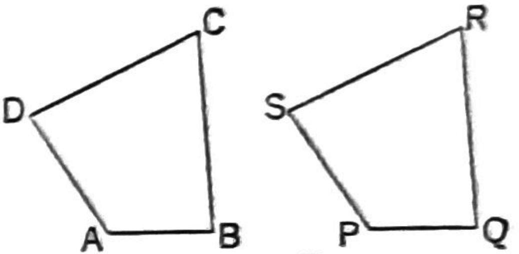
Equilateral Δ
also Equiangular Δ

31. When all the sides are equal, the triangle is called *equilateral.* Here $AB = BC = AC$. When all the angles are equal, the triangle is called *equiangular.* Here $\angle A = \angle B = \angle C$.



Isosceles △

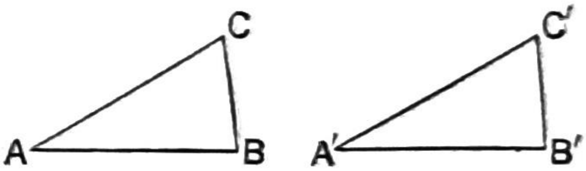
32. When only two sides are equal, the triangle is called *isosceles.* Here $AC = BC$. The equal sides are called *legs.* The third side, AB, is called the *base.* Angles A and B, opposite the legs, are called the *base angles* of isosceles △ ABC.



Congruent Figures

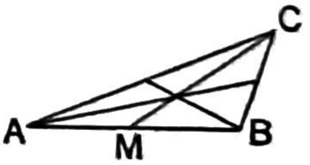
33. If any two geometric figures, such as $ABCD$ and $PQRS$, can be placed one upon the other so that they fit in every part, the figures are said to coincide, and are called *congruent.*

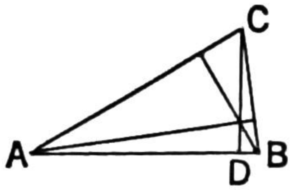
34. Often it is convenient to label corresponding points as is done in this figure. Thus A' corresponds to A, B' to B, and C' to C.



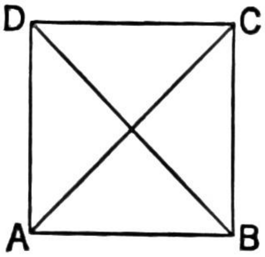
If △ ABC is congruent to △ $A'B'C'$, the *corresponding sides* are AB and $A'B'$, BC and $B'C'$, AC and $A'C'$; and the *corresponding angles* are ∠ A and ∠ A', ∠ B and ∠ B', ∠ C and ∠ C'. Note that corresponding angles of congruent triangles are opposite the correspondingly equal sides, and that corresponding sides are opposite the correspondingly equal angles.

35. The line CM drawn from C to the midpoint of AB is called a *median* of △ ABC. A triangle has three medians.





36. The line CD drawn from C perpendicular to AB is called the *altitude* on AB. A triangle has three altitudes.

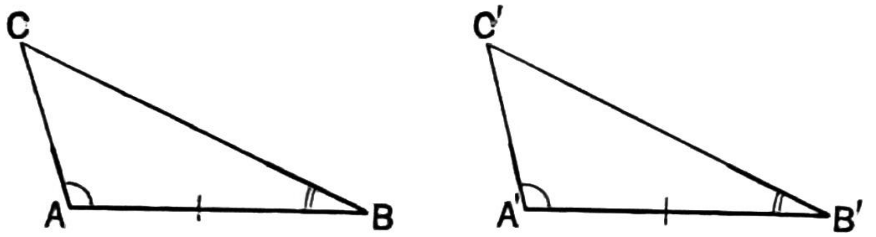


37. $ABCD$ is a *square.* Its four sides are equal, and its four angles are right angles. AC and BD are its *diagonals.*

TRIANGLES

Proposition II. Theorem

38. Two triangles are congruent if two angles and the included side of one are equal respectively to two angles and the included side of the other. (a. s. a. = a. s. a.)



Given $\triangle$ ABC and $A'B'C'$ with $\angle A = \angle A'$, $\angle B = \angle B'$, $AB = A'B'$.

Prove $\triangle ABC \cong \triangle A'B'C'$.

Plan is to place one $\triangle$ *upon the other, and show that they coincide.*

Proof. Place $\triangle ABC$ upon $\triangle A'B'C'$ so that AB falls on its equal $A'B'$ (A on A' and B on B'). As a result of this placing it follows that

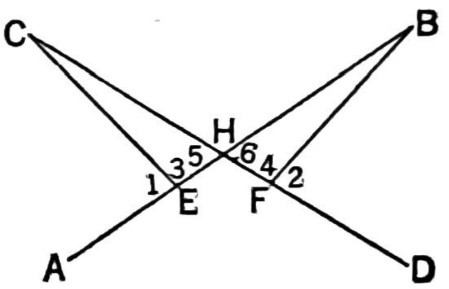
(1) AC must fall along $A'C'$	(1) $\angle A$ is given equal to $\angle A'$.
(2) BC must fall along $B'C'$	(2) $\angle B$ is given equal to $\angle B'$.
(3) $\therefore$ point C must fall on point C'	(3) Two straight lines can intersect in only one point.
(4) $\therefore \triangle ABC \cong \triangle A'B'C'$	(4) We have shown that they fit in every part.
	Q.E.D.

Note: This proposition therefore tells us at once that $AC = A'C'$, $BC = B'C'$, $\angle C = \angle C'$.

39. An *analysis* is a method of procedure whereby we discover what to do in order to prove a statement. Study the following very carefully.

Ex. 27. *Given* AB and CD straight lines, $\angle 1 = \angle 2$, $HE = HF$

Prove $CE = BF$.



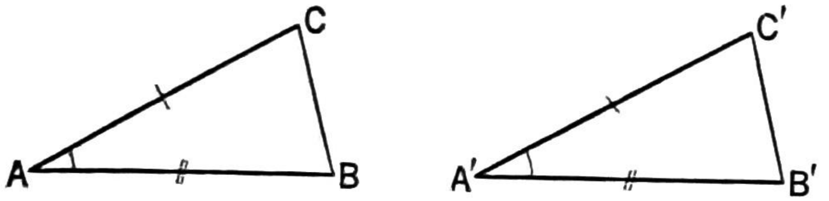
Analysis. We can prove $CE = BF$ *if we can prove they are corresponding parts of congruent* $\triangle$s. *Therefore, we try to prove* $\triangle\ CEH$ *congruent to* $\triangle\ BFH$ *by a. s. a.*

Proof.

(1) $\angle 1 = \angle 2$	(1) Given.
(2) $\therefore \angle 3 = \angle 4$	(2) Supplements of equal angles are equal.
(3) $HE = HF$	(3) Given.
(4) $\angle 5 = \angle 6$	(4) Prop. I.
(5) $\therefore \triangle CEH \cong \triangle BFH$	(5) a. s. a. = a. s. a.
(6) $\therefore CE = BF$	(6) Corresponding parts of congruent $\triangle$s are equal.
	Q.E.D.

Proposition III. Theorem

40. Two triangles are congruent if two sides and the included angle of one are equal respectively to two sides and the included angle of the other. (s. a. s. = s. a. s.)



Given $\triangle$s ABC and $A'B'C'$ with $AB = A'B'$, $AC = A'C'$, $\angle A = \angle A'$.

Prove $\triangle ABC \cong \triangle A'B'C'$.

Plan is to place one $\triangle$ *upon the other, and show that they coincide.*

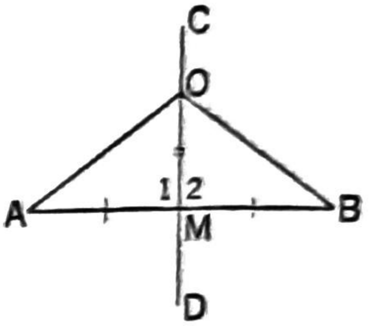
Proof. Place $\triangle ABC$ upon $\triangle A'B'C'$ so that AB falls on its equal $A'B'$ (A on A' and B on B'). As a result of this placing, it follows that

(1) AC must fall along $A'C'$	(1) $\angle A$ is given equal to $\angle A'$.
(2) Point C must fall on point C'	(2) AC is given equal to $A'C'$.
(3) $\therefore BC$ must fall on $B'C'$	(3) Only one straight line can be drawn connecting two points.
(4) $\therefore \triangle ABC \cong \triangle A'B'C'$	(4) We have shown that they fit in every part.

Q.E.D.

Let us apply this theorem to the following example:

Ex. 28. Any point in the perpendicular bisector of a line is equally distant from the ends of the line.



Given $CD \perp AB$, $AM = MB$, and O any point in CD.

Prove $OA = OB$.

Analysis. We can prove $OA = OB$ *if we can prove that the* $\triangle$ OMA *and* OMB *are congruent. This we can do by s. a. s.*

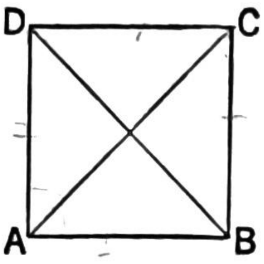
Proof.

(1) $AM = MB$	(1) Given.
(2) $\angle 1 = \angle 2$	(2) All right angles are equal.
(3) $OM = OM$	(3) Identity.
(4) $\therefore \triangle OMA \cong \triangle OMB$	(4) s. a. s. = s. a. s.
(5) $\therefore OA = OB$	(5) Corresponding parts of congruent $\triangle$ are equal. Q.E.D.

Exercises

41. The following seven exercises can be worked by applying one or more of the first three propositions. Remember that lines or angles may be proved equal by proving them corresponding parts of congruent triangles. Before going into the details of a proof, be sure to have in mind a definite plan of procedure. If you are not sure how to proceed, make a careful *analysis*. An analysis consists in saying to yourself "I can prove *a* if I can prove *b*; I can prove *b* if I can prove *c*, and so on till I arrive at something that I can prove." Then reverse your steps.

Ex. 29. The diagonals of a square are equal.



Given $ABCD$ a square, with its diagonals AC and BD.

Prove $AC = BD$.

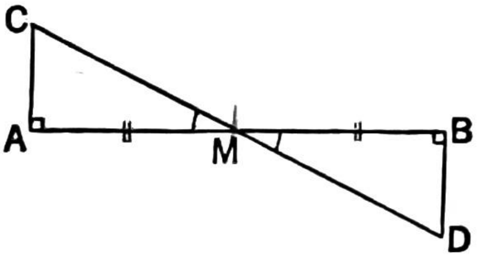
Analysis. We can prove $AC = BD$ *if we can prove that* △s ABC *and* ABD *are congruent. We can prove these triangles congruent if we can show that there is either a s. a. s. condition or an a. s. a. condition. But we can show that there is an s. a. s. condition.* ∴ *we can prove* $AC = BD$.

Proof. In △s ABD and ABC

(1) $AD = BC$	(1) 37.
(2) $AB = AB$	(2) 23.
(3) $\angle DAB = \angle CBA$	(3) 37.
(4) ∴ $\triangle ABD \cong \triangle ABC$	(4) s. a. s. = s. a. s.
(5) ∴ $BD = AC$	(5) Corr. parts of congruent △s are equal.

Q.E.D.

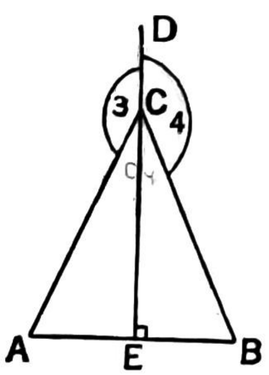
Ex. 30. *Given* M is the midpoint of AB, $AC \perp AB$, $BD \perp AB$, CD is a straight line.



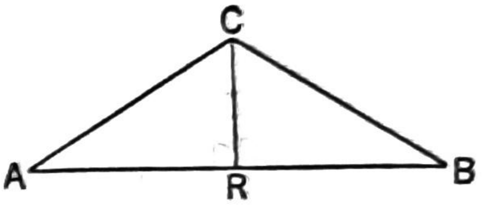
Prove $\angle C = \angle D$, $AC = BD$, $CM = MD$.

Ex. 31. *Given* $\angle 3 = \angle 4$, $DE \perp AB$, DCE a straight line.

Prove $AC = BC$.

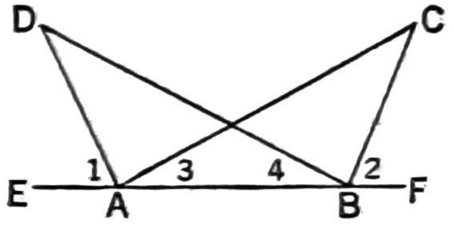


Ex. 32. The bisector of the vertex angle of an isosceles triangle is perpendicular to the base and bisects the base.

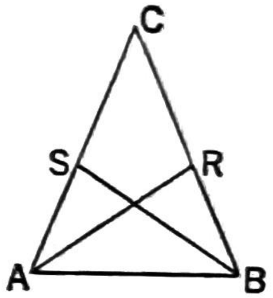


Ex. 33. *Given* $AD = BC$, $\angle 1 = \angle 2$, EF a straight line.

Prove $\angle 3 = \angle 4$.

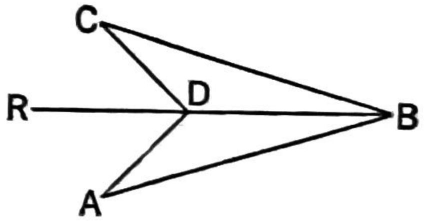


Ex. 34. If two angles of a triangle are equal, the bisectors of these angles are equal.



Ex. 35. BR bisects $\angle ABC$ and $\angle ADC$.

Prove $AB = CB$, and $AD = CD$.



GROUP I

Complete each of the following statements:

(*a*) If one of two vertical angles is 36°, the other is ....

(*b*) If two angles are supplementary, their sum is ....

(*c*) If two angles are complementary, their sum is ....

(*d*) If three equal angles are drawn about a point in a plane, each angle is ....

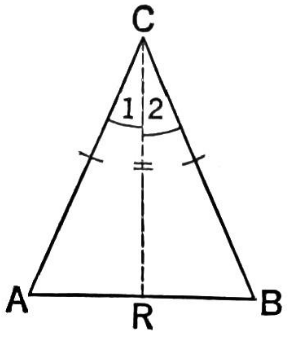
(*e*) If $\angle A = \angle B$, then $4 \angle A = 4 \angle B$ because ....

(*f*) Geometric figures that can be made to coincide in all their parts are said to be ....

(*g*) The parts of a theorem are ....

Proposition IV. Theorem

42. **The base angles of an isosceles triangle are equal.**



Given Isosceles $\triangle ABC$ with $BC = AC$.

Prove $\angle A = \angle B$.

Analysis. We can prove $\angle A = \angle B$ *if we can make them corresponding parts of congruent △s. This suggests that we draw a line from* C *that will give congruent △s. If that line were drawn, the angles at* C *would be equal by corresponding parts, suggesting that we bisect* $\angle C$.

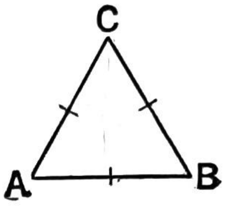
Proof. Bisect $\angle ACB$ by CR. Then

(1) $CA = CB$	(1) Given.
(2) $\angle 1 = \angle 2$	(2) By construction.
(3) $CR = CR$	(3) Identity.
(4) $\therefore \triangle CRA \cong \triangle CRB$	(4) s. a. s. = s. a. s.
(5) $\therefore \angle A = \angle B$	(5) Corresponding parts of congruent △s are equal.

Q.E.D.

43. A *corollary* is a statement that follows very closely as the result of another statement. To prove a corollary, we make use of the theorem to which it has a logically close connection. Note that the truth of a corollary is not taken for granted, but requires proof.

44. *Cor.* An equilateral triangle is also equiangular.



Sug. $AC = BC \quad \therefore \angle A = \angle B$
$AC = AB \quad \therefore \angle B = \angle C$

45. *Remarks:* Very often it is necessary to draw additional lines to help with the proof of a theorem. Such lines are represented by dotted lines, to distinguish them from given lines. Be careful not to put more conditions on a line than are necessary to determine the line. Thus in Prop. IV it would be wrong to say "Draw CR to the middle of AB and so as to bisect $\angle C$." Note that had we said "Draw CR to the middle of AB," we would have had a perfectly allowable construction, but such a construction would be useless because we could neither apply s. a. s. nor a. s. a. Had we said "Draw $CR \perp AB$," we would also have had an allowable construction, but useless.

GROUP II

Tell whether the following statements are true or false, and give reasons:

(*a*) If in the isosceles $\triangle ABC$ points R and S are taken on base AB and situated between A and B so that $AR = BS$, then $\angle CRS = \angle CSR$.

(*b*) ABC and ABD are two isosceles triangles situated on the same side of the base AB. Hence the straight line joining C and D bisects $\angle ADB$.

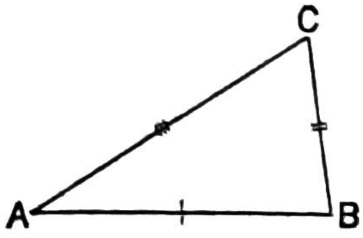
(*c*) If straight lines are drawn connecting the midpoints of the sides of an equilateral triangle, another equilateral triangle is formed.

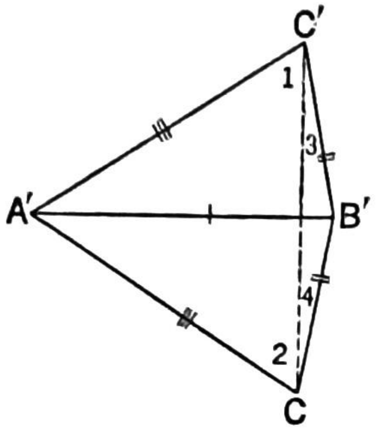
(*d*) The supplement of an angle is always greater than the angle.

(*e*) The difference between the supplement and the complement of an angle is 90°.

Proposition V. Theorem

46. Two triangles are congruent if three sides of one are respectively equal to three sides of the other. (s. s. s. = s. s. s.)





Given $\triangle$s ABC and $A'B'C'$ with $AB = A'B'$, $BC = B'C'$, $AC = A'C'$.

Prove $\triangle\ ABC$ is congruent to $\triangle\ A'B'C'$.

Analysis. We can not place $\triangle\ ABC$ *upon* $\triangle\ A'B'C'$ *and show that they coincide, because nothing is known about the angles. We therefore try to make use of any of the previous propositions. Hence the plan is to use s. a. s.*

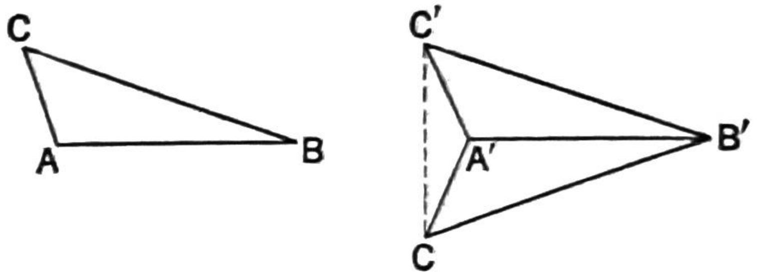
Proof. Place $\triangle\ ABC$ so that AB shall fall upon its equal $A'B'$ (A on A' and B on B'), and vertex C below $A'B'$. Draw $C'C$, thus forming $\triangle$s $CA'C'$ and $CB'C'$. Then

(1) $A'C' = AC\ (A'C)$	(1) Given.
(2) $\therefore \angle 1 = \angle 2$	(2) 42.
(3) $B'C' = BC\ (B'C)$	(3) Given.
(4) $\therefore \angle 3 = \angle 4$	(4) 42.
(5) $\therefore \angle 1 + \angle 3 = \angle 2 + \angle 4$	(5) Ax. 2.
(6) $\therefore \angle A'C'B' = \angle A'CB'$	(6) Ax. 6.
(7) $\therefore \triangle\ A'B'C' \cong \triangle\ A'B'C$	(7) s. a. s. = s. a. s.
(8) $\therefore \triangle\ A'B'C' \cong \triangle\ ABC$	(8) Sub.

Q.E.D.

Exercises

Ex. 36. Prove Prop. V if the given triangles have the shape as indicated in the annexed figure, and are placed as there shown.



Ex. 37. The medians to the legs of an isosceles triangle are equal.

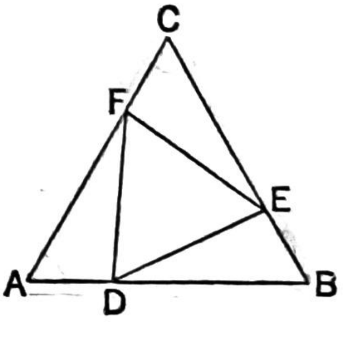
Ex. 38. If the base of an isosceles triangle is trisected (divided into three equal parts), the lines joining the points of division with the vertex are equal.

Ex. 39. *ABC* is an equilateral triangle. *AB* is prolonged to *D*, *BC* to *E*, and *CA* to *F*. If $BD = CE = AF$, and if points *D*, *E*, and *F* are connected by straight lines, prove that $\triangle DEF$ is equilateral.

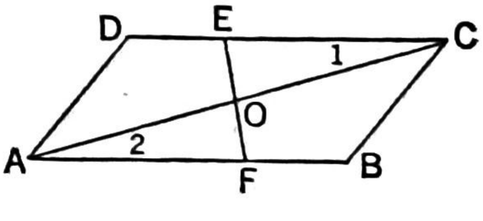
Sug. Prove $\triangle ADF \cong \triangle BDE \cong \triangle CEF$.

Ex. 40. If on the sides of an equilateral $\triangle ABC$ the points *D*, *E*, and *F* are taken so that $AD = BE = CF$, then $\triangle DEF$ is equilateral.

Sug. Prove $\triangle ADF \cong \triangle BDE \cong \triangle CFE$.



Ex. 41. If the opposite sides of a quadrilateral (a four-sided closed figure) are equal, and a line is drawn through the midpoint of a diagonal, terminating in two sides, this line is bisected.

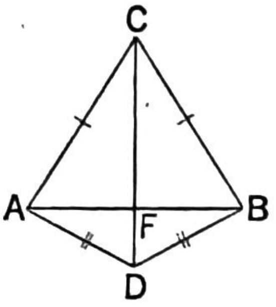


Sug. Prove $\triangle ADC \cong \triangle ABC$, then use $\triangle$ *AOF* and *EOC*.

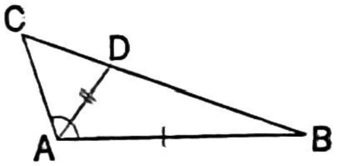
Ex. 42. Two triangles are congruent if two sides and the median to one of the sides are equal respectively to two sides and the corresponding median of the other.

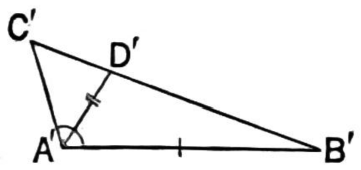
Ex. 43. Two points each equally distant from the ends of a line determine the perpendicular bisector of the line.

Sug. First use $\triangle$ *ACD* and *BCD*, then $\triangle$ *ACF* and *BCF*.

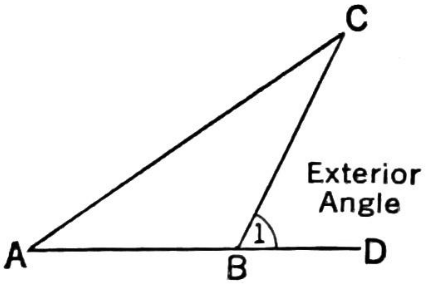


Ex. 44. Prove that two triangles are congruent if a side, an adjacent angle, and the bisector of that angle are equal respectively to the corresponding parts in the other.

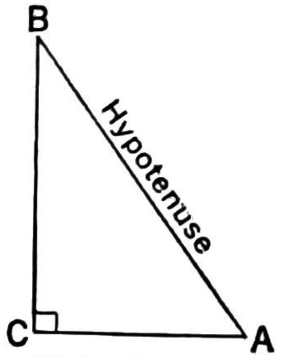




47. *Def.* $\angle 1$ is called an *exterior angle* of $\triangle ABC$. Note that it is formed by any side (*BC*) and another side prolonged. (*AB* is prolonged.)



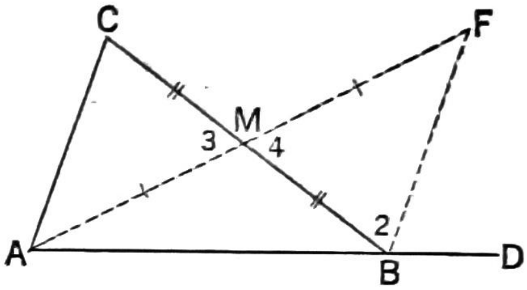
48. *Def.* $\triangle BCA$ has a right angle *C*. It is therefore called a *right triangle*. *AB*, the side opposite the right angle, is called the *hypotenuse*. *CB* and *CA* are called the *legs* of the triangle.



Right Triangle

Proposition VI. Theorem

49. An exterior angle of a triangle is greater than either opposite interior angle.



Given $\angle\ CBD$ an exterior angle of $\triangle\ ABC$.

Prove $\angle\ CBD > \angle\ C$, also $\angle\ CBD > \angle\ CAB$.

Analysis. We can prove $\angle\ CBD > \angle\ C$ *if we can show that* $\angle\ C$ *is equal to an angle which is only a part of* $\angle\ CBD$. *We can get* $\angle\ C = \angle\ 2$ *by making them corresponding parts of congruent* $\triangle$s. *If we imagine the* $\triangle$s *drawn,* CM *would equal* MB, *and* AM *would equal* MF, *suggesting that we connect* A *with the midpoint of* CB *and prolong* AM *its own length.*

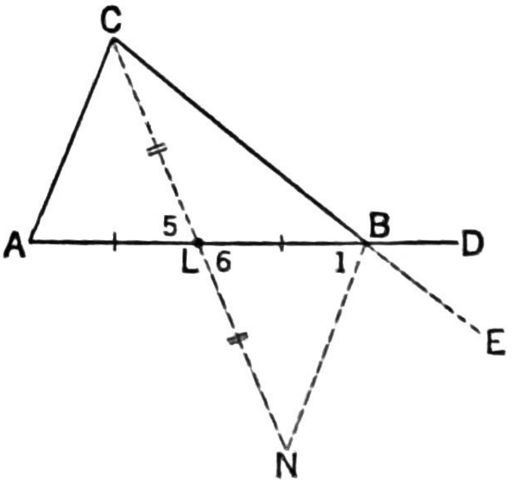
Plan is to prove $\angle\ C = \angle\ 2$ *(which is only a part of* $\angle\ CBD$*).*

Proof. Let M be the midpoint of BC. Draw AM, and prolong it its own length to F. Draw BF. Then

(1) $\angle\ CBD > \angle\ 2$	(1) The whole is greater than any of its parts.
(2) but $\angle\ C = \angle\ 2$	(2) Corresponding parts of congruent triangles are equal. ($\triangle\ AMC \cong \triangle\ BMF$ by s. a. s.)
(3) $\therefore\ \angle\ CBD > \angle\ C$	(3) Substitution.

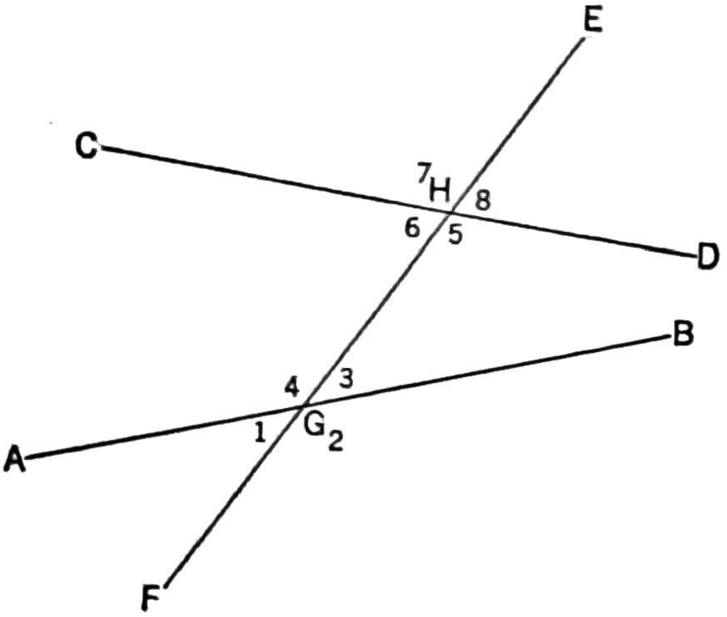
In like manner we can prove $\angle CBD > \angle CAB$. Thus

$\angle ABE > \angle 1$
but $\angle 1 = \angle A$
$\therefore \angle ABE > \angle A$
but $\angle ABE = \angle CBD$
$\therefore \angle CBD > \angle A$
Q.E.D.



Angles Formed by Transversal

50. In this drawing, AB and CD are crossed by EF at G and H respectively. EF is called a *transversal* of AB and CD. Note that eight angles are formed:



$\measuredangle$ 1, 2, 7, 8 are called *exterior* angles.
$\measuredangle$ 3, 4, 5, 6 are called *interior* angles.

$$\left.\begin{array}{r}\angle 6 \text{ and } \angle 3\\ \text{or } \angle 5 \text{ and } \angle 4\end{array}\right\} \text{are called } \textit{alternate interior} \text{ angles.}$$

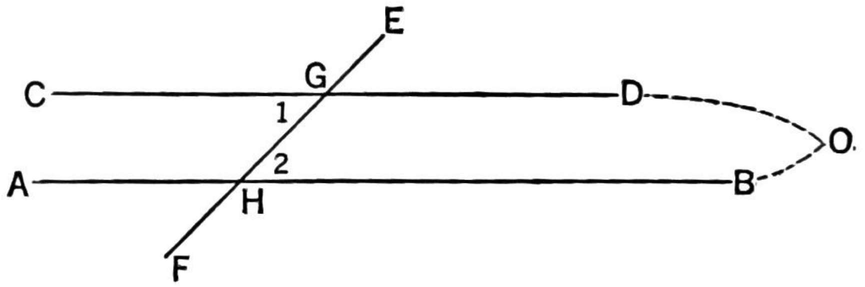
$$\left.\begin{array}{r}\angle 7 \text{ and } \angle 2\\ \text{or } \angle 8 \text{ and } \angle 1\end{array}\right\} \text{are called } \textit{alternate exterior} \text{ angles.}$$

$$\left.\begin{array}{r}\angle 8 \text{ and } \angle 3\\ \text{or } \angle 7 \text{ and } \angle 4\\ \text{or } \angle 2 \text{ and } \angle 5\\ \text{or } \angle 1 \text{ and } \angle 6\end{array}\right\} \text{are called } \textit{corresponding} \text{ angles.}$$

PARALLEL LINES

Proposition VII. Theorem

51. If two straight lines are cut by a transversal, and if the alternate interior angles are equal, the lines are parallel.



Given AB and CD two straight lines cut by transversal EF at H and G respectively, and that $\angle 1 = \angle 2$.

Prove $AB \parallel CD$.

Analysis. We can prove two lines in the same plane $\parallel$, *if we can prove it impossible for them to meet.*

Plan is to show that if the lines could meet, we would have an absurd state of affairs, viz. a contradiction of what is given.

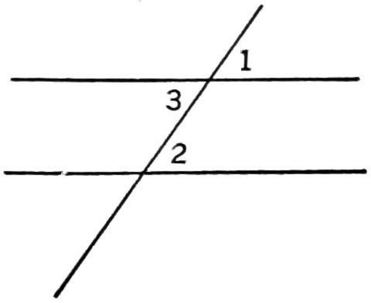
Proof. Suppose AB and CD are not $\parallel$. In that case they will meet if sufficiently prolonged, thus forming $\triangle GHO$. Then

(1) $\angle 1 > \angle 2$	(1) Prop. VI.
(2) but $\angle 1 = \angle 2$	(2) Given.
(3) $\therefore CD$ cannot meet AB	(3) The supposition that they can meet leads to a contradiction of what is given.
(4) $\therefore CD \parallel AB$	(4) Def. of $\parallel$s.

Q.E.D.

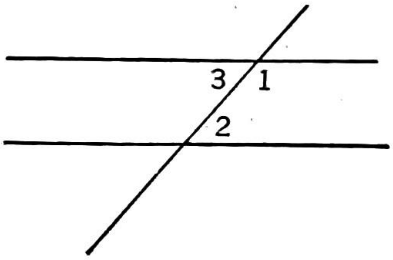
52. The method of proof here used is called *indirect*, whereas the method of the first six propositions is *direct*. When we use the indirect method we suppose that the very opposite of what we have to prove is true, and then show that this supposition leads to an absurdity. We therefore conclude that what we originally were asked to prove must be true.

53. *Cor.* 1. If two straight lines are cut by a transversal, and if the corresponding angles are equal, then the lines are parallel.



Outline of proof. $\angle 1 = \angle 2$
$\angle 1 = \angle 3$
$\therefore \angle 2 = \angle 3$
$\therefore$ lines are $\parallel$

54. *Cor.* 2. If two straight lines are cut by a transversal, and if the interior angles on the same side of the transversal are supplementary, then the lines are parallel.



Outline of proof. $\angle 2$ is sup. of $\angle 1$
$\angle 3$ is sup. of $\angle 1$
$\therefore \angle 2 = \angle 3$
$\therefore$ lines are $\parallel$

55. *Cor.* 3. Two straight lines perpendicular to the same straight line are parallel.

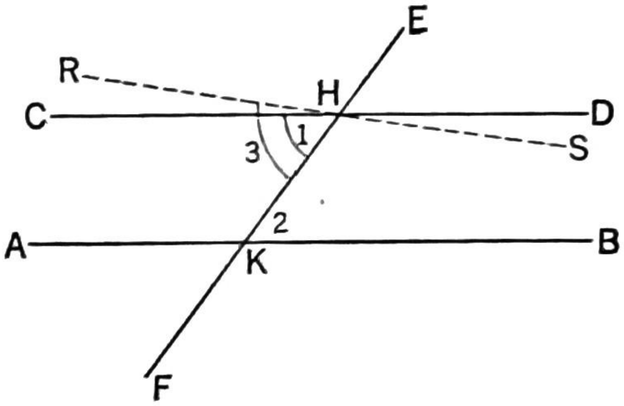
Tell whether the following statements are true or false, and give reasons:

(*a*) $\angle DBC$ is an exterior angle of $\triangle ABC$, $\angle A = 75°$, $\angle B = 30°$. $\therefore \angle DBC = 75°$.

(*b*) In $\triangle ABC$, $AC = BC$. P is the midpoint of AC, Q is the midpoint of BC. Therefore $\angle CPQ = \angle CQP$.

Proposition VIII. Theorem (*Converse of Prop. VII*)

56. If two parallel lines are cut by a transversal, the alternate interior angles are equal.



Given $\parallel$s AB and CD cut by transversal EF at K and H respectively.

Prove $\angle 1 = \angle 2$.

Plan is to show that the supposition that $\angle 1$ *does not equal* $\angle 2$ *leads to an absurdity.*

Proof. $\angle 1$ either equals $\angle 2$ or it does not. If we suppose that it does not, then let us draw RS through H, making $\angle 3 = \angle 2$. Then

(1) $\angle 3 = \angle 2$	(1) Const.
(2) $\therefore RS \parallel AB$	(2) 51.
(3) but $CD \parallel AB$	(3) Given.
(4) $\therefore$ we arrive at an absurdity, viz. a contradiction of the $\parallel$ axiom.	(4) 3 II.
(5) $\therefore \angle 1 = \angle 2$	(5) The only possibility left.

Q.E.D.

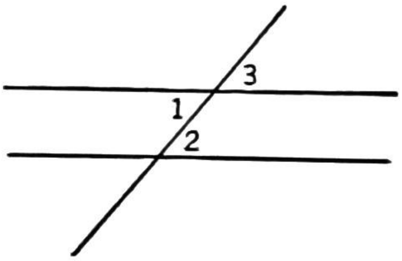
57. Compare the statement of Prop. VII with that of Prop. VIII, and note that what is given in Prop. VII has to be proved in Prop. VIII, and what has to be proved in

Prop. VII is given in Prop. VIII. When two propositions are related to each other in this manner, one is said to be the *converse* of the other. The converse of a proposition may or may not be true, and therefore requires proof. The following is an example of a proposition whose converse is not true:

(*a*) If two angles are right angles, the two angles are equal.

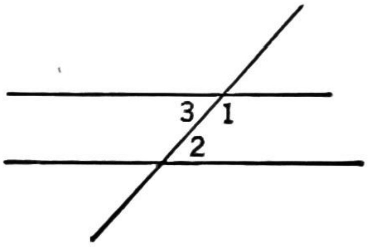
(*b*) Conversely, if two angles are equal, the two angles are right angles. Obviously (*b*) is not true, because two angles can be equal without being right angles.

58. *Cor.* 1. If two parallels are cut by a transversal, the corresponding angles are equal.



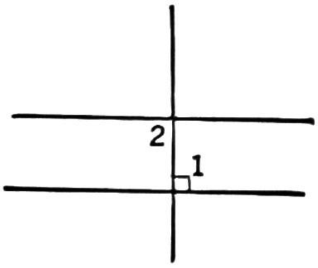
Outline of proof. $\angle 1 = \angle 2$, $\angle 3 = \angle 1$, $\therefore \angle 3 = \angle 2$

59. *Cor.* 2. If two parallels are cut by a transversal, the interior angles on the same side of the transversal are supplementary.



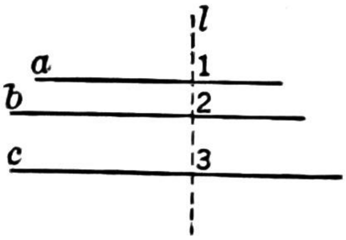
Outline of proof. $\angle 1 + \angle 3 = 180°$, but $\angle 3 = \angle 2$, $\therefore \angle 1 + \angle 2 = 180°$.

60. *Cor.* 3. If a line is perpendicular to one of two parallels, it is perpendicular to the other also.



Outline of proof. $\angle 1 = \angle 2$, but $\angle 1 = 90°$, $\therefore \angle 2 = 90°$.

61. *Cor.* 4. If two lines are parallel to a third line, they are parallel to each other.



Proof. Given $a \parallel c$, $b \parallel c$. *Prove* $a \parallel b$

Draw $l \perp c$, then $\angle 3 = 90°$;

but $\angle 1 = \angle 3$, $\therefore \angle 1 = 90°$.

In like manner prove $\angle 2 = 90°$

$\therefore \angle 1 = \angle 2$, $\therefore a \parallel b$

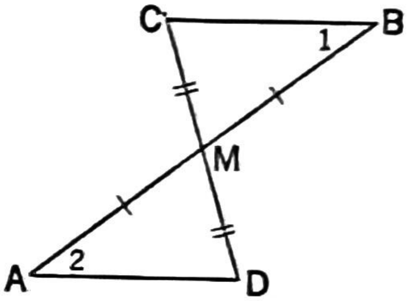
Ex. 45. Prove this corollary, by supposing that a and b are not parallel.

62. *Remark:* Lines may be proved parallel by proving a pair of alternate interior angles equal, or a pair of corresponding angles equal, or the interior angles on the same side of the transversal supplementary.

Exercises

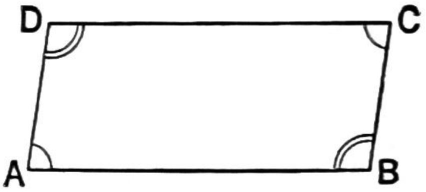
Ex. 46. If AB and CD bisect each other at M, prove that $AD \parallel CB$.

Sug. Prove $\angle 1 = \angle 2$



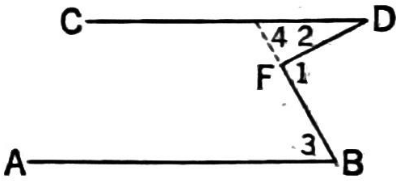
Ex. 47. If $\angle A + \angle B + \angle C + \angle D = 360°$ and $\angle A = \angle C$, $\angle B = \angle D$, prove that $AB \parallel DC$.

Sug. Prove $\angle A + \angle D = 180°$



Ex. 48. Given $\angle 1 = \angle 2 + \angle 3$. Prove $AB \parallel CD$.

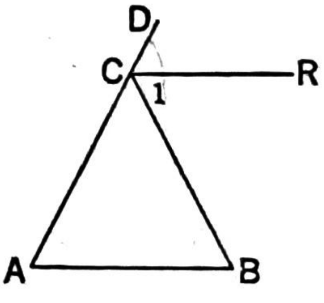
Sug. Prove $\angle 3 = \angle 4$



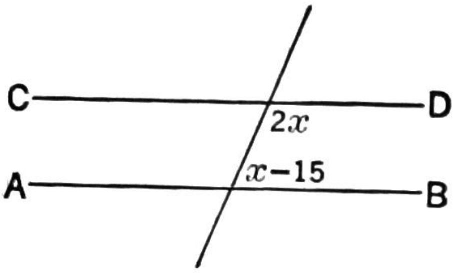
Ex. 49. An exterior angle of a triangle equals the sum of the two opposite interior angles.

Ex. 50. If one of the legs of an isosceles triangle is extended through the vertex, and the exterior angle thus formed is bisected, prove that this bisector is parallel to the base.

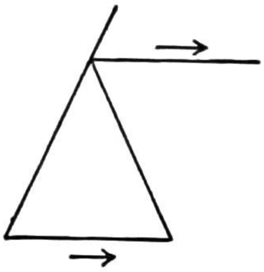
Sug. $\angle DCB = 2 \angle B \quad \therefore \angle 1 = \angle B$, etc.



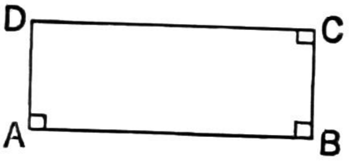
Ex. 51. If $AB \parallel CD$, and the indicated angles are $2x$ degrees and $(x - 15)$ degrees, find the value of x.



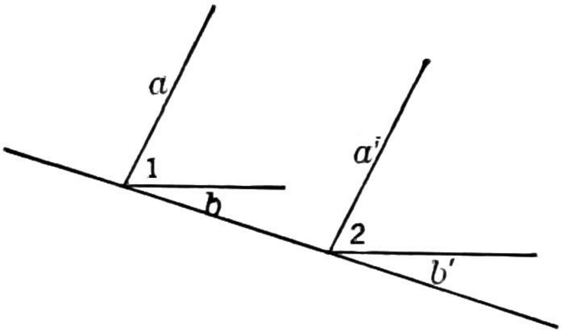
Ex. 52. If a line is drawn through the vertex of an isosceles triangle parallel to the base, it bisects the exterior angle at the vertex.



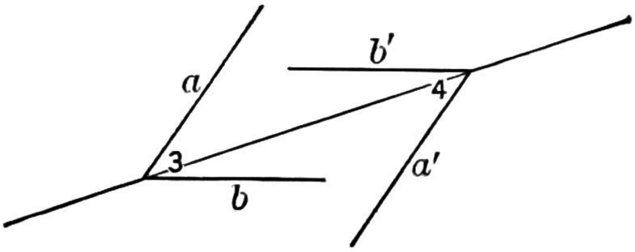
Ex. 53. If in the annexed figure ∠s A, B, C are right angles, ∠ D is also a right angle.



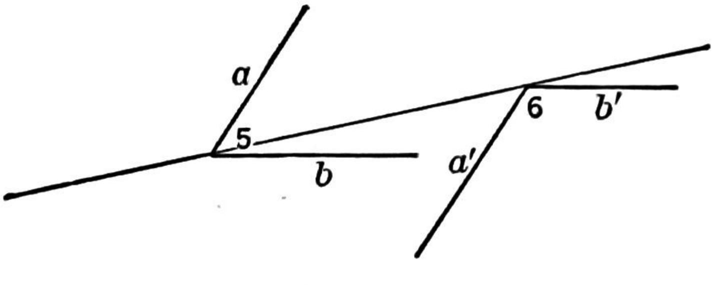
63. Consider ∠s 1 and 2 where $a \parallel a'$ and $b \parallel b'$. The sides of these angles are said to *extend in the same direction from the vertices,* because they are on the same side of a line joining the vertices.



Now consider ∠s 3 and 4 where $a \parallel a'$ and $b \parallel b'$. Here both pairs of ∥s *extend in opposite directions from the vertices,* because they are on opposite sides of the line joining the vertices.

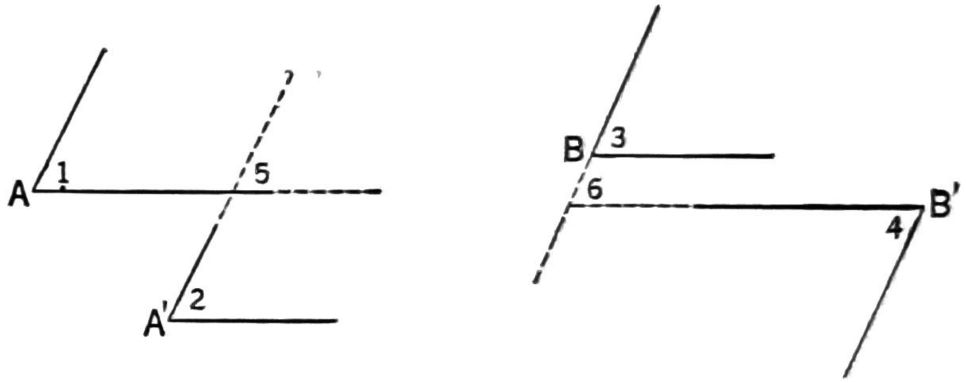


In the case of ∠s 5 and 6 where $a \parallel a'$ and $b \parallel b'$, b and b' extend in the same direction from the vertices, but a and a' extend in opposite directions from the vertices.



Proposition IX. Theorem

64. If two angles have their sides respectively parallel, and if both pairs of parallels extend either in the same direction or in opposite directions from the vertices, the angles are equal.



Given ∠s 1 and 2 with their sides respectively ∥ and extending in the same direction from A and A', and ∠s 3 and 4 with their sides respectively ∥ and extending in opposite directions from B and B'.

Prove ∠ 1 = ∠ 2, ∠ 3 = ∠ 4.

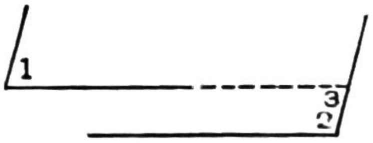
Plan is to get ∥*s cut by a transversal, and to prove the given* ∠s *equal to the same* ∠.

Proof. Extend the sides so as to form ∠ 5 in one case, and ∠ 6 in the other.

(1) ∠ 1 = ∠ 5	(1) Corr. ∠s of ∥s are equal.
(2) ∠ 2 = ∠ 5	(2) Same reason.
(3) ∴ ∠ 1 = ∠ 2	(3) Things equal to the same thing are equal to each other.
(1) ∠ 3 = ∠ 6	(1) Corr. ∠s of ∥s are equal.
(2) ∠ 4 = ∠ 6	(2) Alt. int. ∠s of ∥s are equal.
(3) ∴ ∠ 3 = ∠ 4	(3) Things equal to the same thing are equal to each other.

Q.E.D.

65. *Cor.* If two angles have their sides respectively parallel, and if one pair of ||s extends in the same direction from the vertices and the other pair extends in opposite directions, then the angles are supplementary.

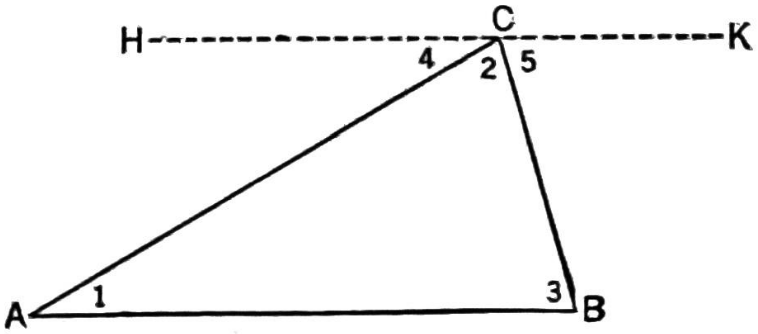


Proof. $\angle 2 + \angle 3 = 180°$, but $\angle 1 = \angle 3$, $\therefore \angle 1 + \angle 2 = 180°$.

TRIANGLES

Proposition X. Theorem

66. The sum of all the angles of a triangle is equal to a straight angle.



Given $\triangle ABC$.

Prove $\angle 1 + \angle 2 + \angle 3 =$ a st. $\angle$.

Analysis. We can prove $\angle 1 + \angle 2 + \angle 3 =$ *one st.* $\angle$ *if we can transfer these angles about one point on the same side of a st. line. This transfer can be accomplished if we get ||s cut by a transversal, suggesting that through any vertex we draw a line || opposite side.*

Proof. Through C draw $HK \parallel AB$. Then

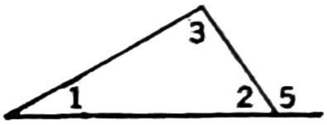
(1) $\angle 4 + \angle 2 + \angle 5 =$ a st. $\angle$	(1) The whole equals the sum of its parts, and the whole $\angle HCK$ is a straight $\angle$.

(2) but $\angle 1 = \angle 4$	(2) Alt. int. $\measuredangle$ of $\parallel$s are equal.
(3) and $\angle 3 = \angle 5$	(3) Same reason.
(4) $\therefore \angle 1 + \angle 2 + \angle 3 = \text{a st. } \angle$	(4) Substitution.

Q.E.D.

67. *Cor.* 1. An exterior angle of a triangle equals the sum of the two opposite interior angles.

Proof. $\angle 1 + \angle 2 + \angle 3 = 180°$
but $\angle 2 + \angle 5 = 180°$
$\therefore \angle 2 + \angle 5 = \angle 1 + \angle 2 + \angle 3$
$\therefore \angle 5 = \angle 1 + \angle 3$



68. *Cor.* 2. In a triangle there can be at most one right angle or one obtuse angle.

69. *Cor.* 3. If two triangles have two angles of one equal respectively to two angles of the other, the third angles are equal.

Sug. If equals are taken from equals, the remainders are equal.

70. *Cor.* 4. Two right triangles are congruent if the hypotenuse and an acute angle of one are equal respectively to the hypotenuse and an acute angle of the other.

Sug. Show that the $\triangle$ really have a. s. a. = a. s. a.

71. *Cor.* 5. Two right triangles are congruent if a leg and an acute angle of one are equal to the corresponding leg and acute angle of the other.

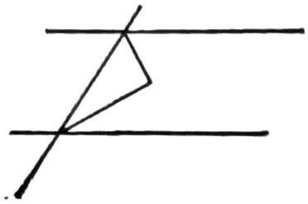
Sug. Show that the $\triangle$ really have a. s. a. = a. s. a.

72. *Cor.* 6. Each angle of an equilateral triangle is 60°.

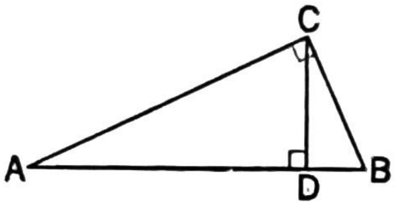
Exercises

Ex. 54. Two triangles are congruent if two angles and the side opposite one of them are equal respectively to two angles and the corresponding side of the other.

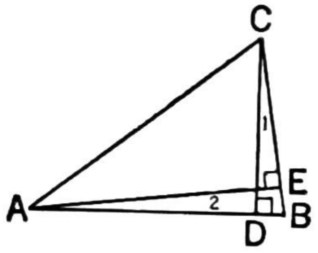
Ex. 55. The bisectors of two interior angles on the same side of a transversal to two parallel lines are perpendicular to each other.



Ex. 56. If CD is the altitude upon the hypotenuse of rt. △ ABC, prove ∠ ACD = ∠ B.

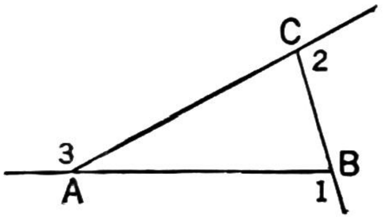


Ex. 57. If CD and AE are altitudes of △ ABC, prove ∠ 1 = ∠ 2.

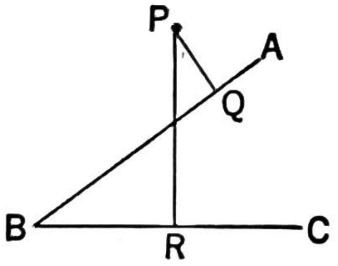


Ex. 58. The altitudes upon the legs of an isosceles triangle are equal.

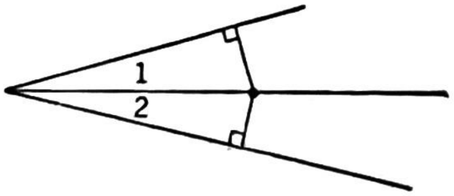
Ex. 59. ABC is any triangle. The sides are prolonged in succession. What is the sum of ∠ 1 + ∠ 2 + ∠ 3?



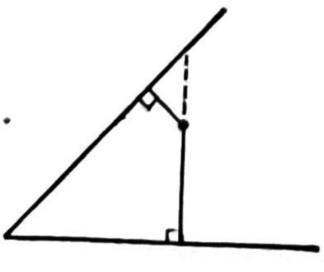
Ex. 60. If from any point outside an acute angle perpendiculars are drawn to the sides of the angle, the angle between these perpendiculars is equal to the given angle.



Ex. 61. Any point in the bisector of an angle is equally distant from the sides of the angle.

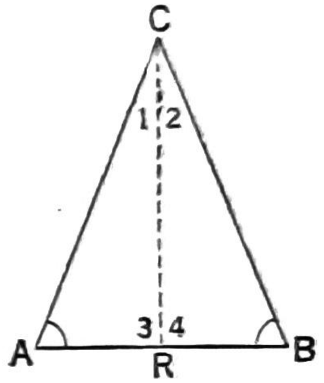


Ex. 62. If from any point inside an acute angle perpendiculars are drawn to the sides of the angle, the angle between these perpendiculars is the supplement of the given angle.



Proposition XI. Theorem (*Converse of Prop. IV*)

73. If two angles of a triangle are equal, the sides opposite these angles are equal.



Given $\triangle ABC$ with $\angle A = \angle B$.

Prove $BC = AC$.

Analysis. We can prove $BC = AC$ *if we can prove that they are corresponding parts of congruent* $\triangle$s. *If we had such* $\triangle$s *then* $\angle 1$ *would equal* $\angle 2$, *suggesting that we bisect* $\angle C$.

Proof. Bisect $\angle C$ by the line CR. Then

(1) $\angle 1 = \angle 2$	(1) Const.
(2) $\angle A = \angle B$	(2) Given.
(3) $\therefore \angle 3 = \angle 4$	(3) 69.
(4) $CR = CR$	(4) Iden.
(5) $\therefore \triangle ARC \cong \triangle BRC$	(5) a. s. a. = a. s. a.
(6) $\therefore AC = BC$	(6) Corr. parts of congruent $\triangle$s are equal.

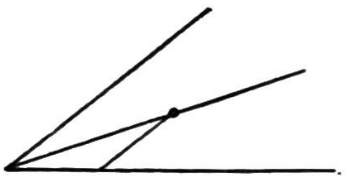
Q.E.D.

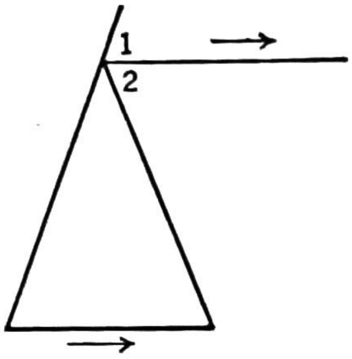
Remark: We might have drawn a line from $C \perp AB$, instead of the construction used above.

74. *Cor.* An equiangular triangle is also equilateral.

Exercises

Ex. 63. If from any point in the bisector of an angle a line is drawn parallel to one of the sides of the angle, an isosceles triangle is formed.

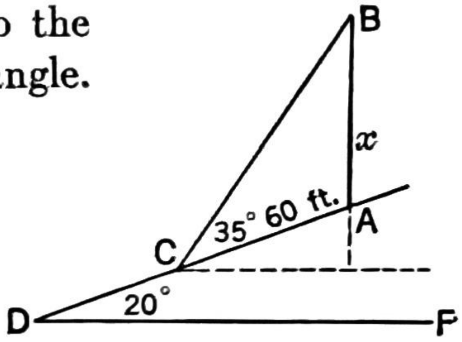




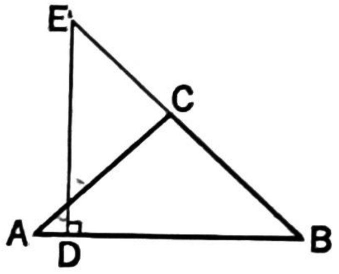
Ex. 64. If the bisector of an exterior angle of a triangle is parallel to the base, the triangle is isosceles.

Ex. 65. If a line cuts the legs of an isosceles triangle and is parallel to the base, it forms another isosceles triangle.

Ex. 66. A tower AB stands on ground that is inclined 20° to a horizontal line DF. At a point C, 60 ft. from A, $\angle ACB = 35°$. Find the height of the tower.



Ex. 67. Prove that if a perpendicular is erected at any point in the base of an isosceles triangle, meeting one leg and the other leg produced, another isosceles triangle is formed external to the given triangle.



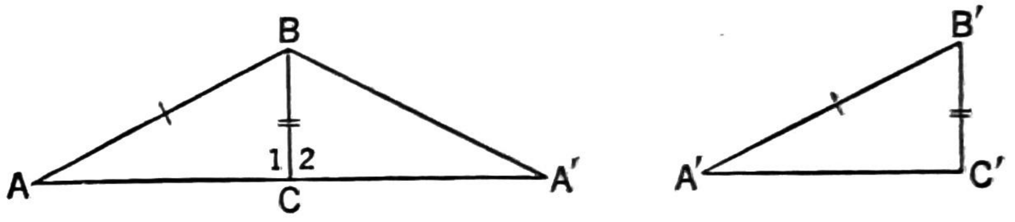
Complete each of the following statements:

(*a*) In an isosceles right triangle each base angle is . . . degrees.

(*b*) One acute angle of a right triangle is four times the other acute angle. Hence the smaller acute angle is . . . degrees.

Proposition XII. Theorem

75. Two right triangles are congruent if the hypotenuse and a leg of one are equal respectively to the hypotenuse and a leg of the other.



Given right triangles ABC and $A'B'C'$, with $AB = A'B'$, $BC = B'C'$.

Prove $\triangle ABC \cong \triangle A'B'C'$.

Analysis. We can not place one triangle upon the other and show that they coincide, because we do not know whether the corresponding acute angles are equal. We therefore must make use of something already proved. Therefore the

Plan is to place the triangles next to each other, and use 70.

Proof. Place $\triangle A'B'C'$ so that $B'C'$ coincides with its equal BC, and A' and A fall on opposite sides of BC. Then

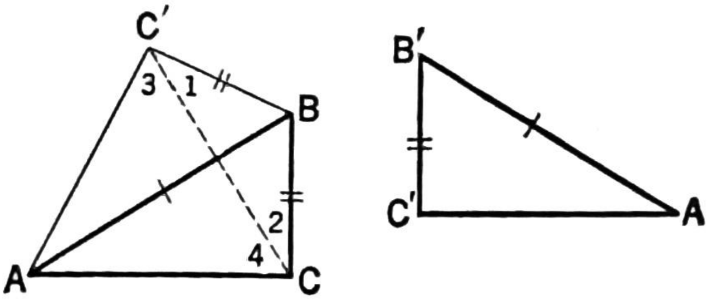
(1) ACA' is a straight line.	(1) $\angle ACA' = \angle 1 + \angle 2 = 90° + 90° = 180°$.
(2) $\therefore ABA'$ is a triangle.	(2) Def. of a $\triangle$.
(3) $AB = A'B' \quad (A'B)$	(3) Given.
(4) $\therefore \angle A = \angle A'$	(4) 42.
(5) $\therefore \triangle ABC \cong \triangle A'BC$	(5) 70.
(6) $\therefore \triangle ABC \cong \triangle A'B'C'$	(6) Sub.

Q.E.D.

Prove the above proposition if the equal legs are AC and $A'C'$ instead of BC and $B'C'$.

Exercises

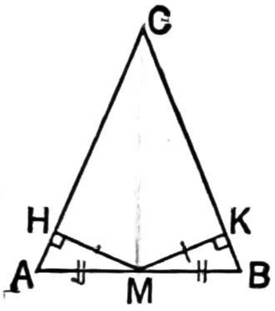
Ex. 68. Prove Prop. XII if the given triangles are placed as indicated in the annexed figure.



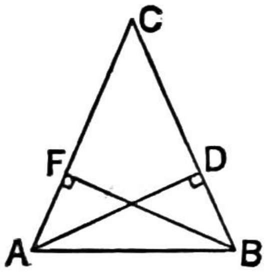
Outline of proof. $\angle 1 = \angle 2$, $\therefore \angle 3 = \angle 4$, $\therefore AC = AC'$, $\therefore \triangle ABC \cong \triangle ABC'$, etc.

Ex. 69. The altitude to the base of an isosceles triangle bisects the base and also the angle at the vertex.

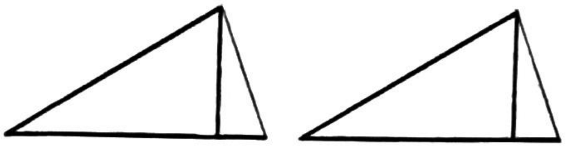
Ex. 70. If the perpendiculars from the midpoint of one side of a triangle upon the other two sides are equal, the triangle is isosceles.



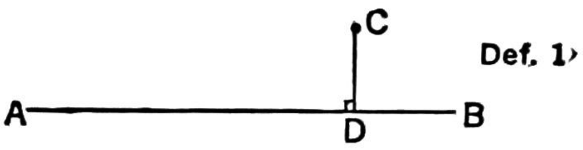
Ex. 71. If two altitudes of a triangle are equal, the triangle is isosceles.



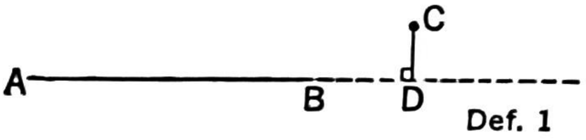
Ex. 72. Two triangles are congruent if two sides and the altitude to one of the sides are respectively equal to two sides and the corresponding altitude.



76. *Def.* (1) The distance of a given point from a given line is the length of the perpendicular from the point to the line. Thus if $CD \perp AB$, CD is called the distance of C from AB.



Def. 1›



Def. 1

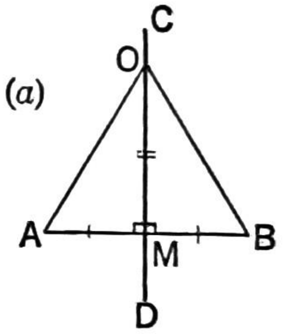
(2) *Equidistant* means equally distant.

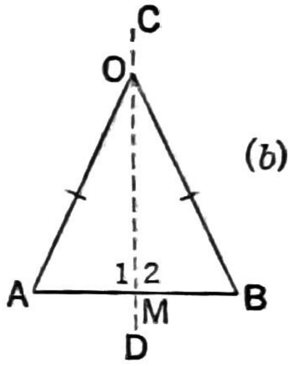
PERPENDICULAR BISECTOR

Proposition XIII. Theorem

77. **(*a*) Every point in the perpendicular bisector of a line is equidistant from the ends of the line, and conversely**

(*b*) Every point equidistant from the ends of a line lies in the perpendicular bisector of the line.





(*a*) *Given* CD the perpendicular bisector of AB, O any point in CD.

Prove $OA = OB$. (See ex. 28.)

(*b*) *Given* line AB, and point O so situated that $OA = OB$.

Prove O is in the perpendicular bisector of AB.

Analysis. There are two possible ways to proceed. We might draw a line connecting O *with the middle of* AB, *and then show that this line must also be* $\perp AB$. *Or, we might draw a line from* $O \perp AB$, *and then show that this line must also bisect* AB.

Proof. Take M the midpoint of AB, draw OM.

(1) $\therefore \triangle OMA \cong \triangle OMB$	(1) s. s. s. = s. s. s.
(2) $\therefore \angle 1 = \angle 2$	(2) Why?
(3) $\therefore \angle 1 = 90°$	(3) 12.
(4) $\therefore CD \perp AB$	(4) 13.

Q.E.D.

Let the pupil also prove (*b*) by the second method of the analysis.

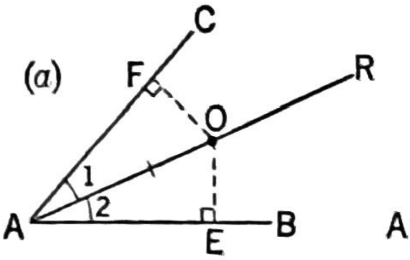
78. *Cor.* Two points each equidistant from the ends of a line determine the perpendicular bisector of the line.

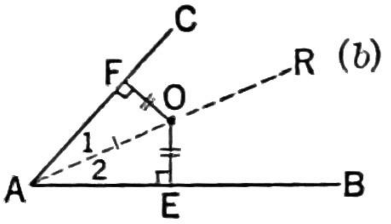
ANGLE BISECTOR

Proposition XIV. Theorem

79. (*a*) Every point in the bisector of an angle is equidistant from the sides of the angle, and conversely

(*b*) Every point within an angle equidistant from the sides of the angle lies in the bisector of that angle.





(*a*) *Given* AR the bisector of $\angle BAC$, O any point in AR.

Prove O is equidistant from AB and AC.

Analysis. Since the distance of a point from a line is the length of the perpendicular from the point to the line, we must draw $OF \perp AC$ *and* $OE \perp AB$. *We can prove* $OE = OF$ *by proving them corresponding parts of congruent* $\triangle$.

Proof. Draw $OE \perp AB$, and $OF \perp AC$.

(1) $\angle 2 = \angle 1$	(1) Given.
(2) $AO = AO$	(2) Iden.
(3) $\therefore \triangle AOE \cong \triangle AOF$	(3) 70.
(4) $\therefore OE = OF$	(4) Corr. parts of congruent $\triangle$ are equal.

Q.E.D.

(*b*) *Given* $\angle BAC$, and O any point within the angle so situated that $OE = OF$ (where $OE \perp AB$ and $OF \perp AC$).

Prove O lies in the bisector of $\angle BAC$.

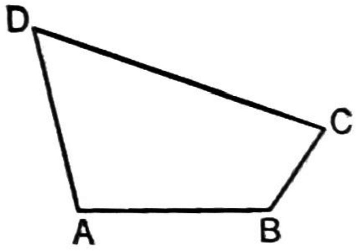
Analysis. If we can prove that the line connecting O *with* A *bisects* $\angle BAC$, *then our theorem is established. This can be done by congruent* △.

Proof. Draw AO. Then

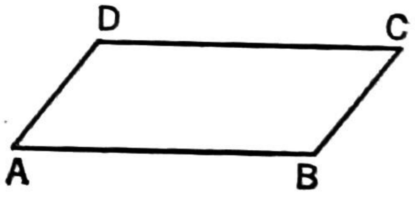
(1) $OE = OF$	(1) Given.
(2) $AO = AO$	(2) Iden.
(3) $\therefore \triangle AOE \cong \triangle AOF$	(3) 75.
(4) $\therefore \angle 2 = \angle 1$	(4) Corr. parts of congruent △ are equal.
(5) $\therefore AO$ is the bisector of $\angle BAC$.	(5) Def. of bisector.

Q.E.D.

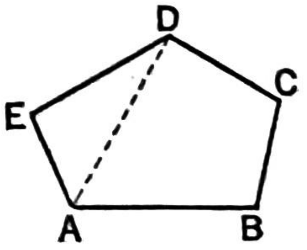
80. A *quadrilateral* is a closed figure bounded by four straight lines, all in the same plane. Thus, $ABCD$ is a quadrilateral.



81. A *parallelogram* is a quadrilateral whose opposite sides are parallel in pairs. Thus, $ABCD$ is a parallelogram because $AB \parallel DC$ and $AD \parallel BC$.



82. A *polygon* is a closed figure bounded by three or more straight lines, all in the same plane. Thus, $ABCDE$ is a polygon of five sides.

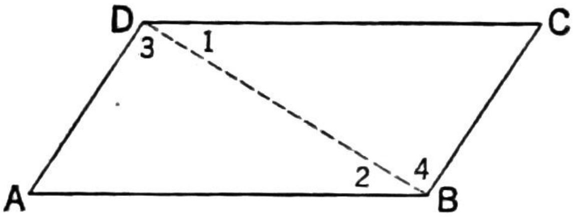


83. A *diagonal* is a straight line joining any two non-consecutive vertices of a polygon. E.g. AD is a diagonal of polygon $ABCDE$. How many diagonals can be drawn in a polygon of five sides?

PARALLELOGRAMS

Proposition XV. Theorem

84. The opposite sides of a parallelogram are equal, and the opposite angles are equal.



Given ▱ $ABCD$.

Prove $AB = DC$, $AD = BC$, $\angle A = \angle C$, $\angle ADC = \angle ABC$.

Plan is to get congruent $\triangle$s.

Proof. Draw DB. Then

(1) $\angle 1 = \angle 2$	(1) 56.
(2) $\angle 3 = \angle 4$	(2) 56.
(3) $DB = DB$	(3) Iden.
(4) $\therefore \triangle ABD \cong \triangle BCD$	(4) a. s. a. = a. s. a.
(5) $\therefore \begin{cases} AB = DC \\ AD = BC \\ \angle A = \angle C \end{cases}$	(5) Corr. parts of congruent $\triangle$s are equal.
(6) also $\angle ADC = \angle ABC$	(6) Ax. 2.

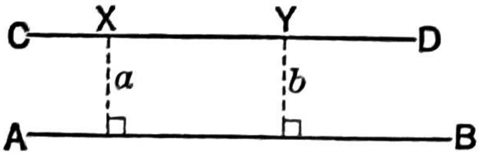
Q.E.D.

85. *Cor.* 1. A diagonal divides a parallelogram into two congruent triangles.

86. *Cor.* 2. Parallels included between parallels are equal.

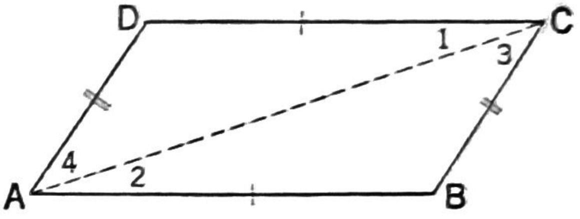
87. *Cor.* 3. Two parallel lines are everywhere equidistant.

Outline of proof. From *any* two points in CD, such as X and Y, draw a and b $\perp AB$. Then $a \parallel b$ $\therefore a = b$.



Proposition XVI. Theorem

88. If the opposite sides of a quadrilateral are equal in pairs, the figure is a parallelogram.



Given quad. $ABCD$, with $AB = DC$ and $AD = BC$.

Prove $ABCD$ is a parallelogram.

Analysis. We can prove $AB \parallel DC$ *and* $AD \parallel BC$ *if we can prove the alt. int. angles equal. We can prove these* $\angle$s *equal if we can prove that they are corr. parts of congruent* $\triangle$s. *We can get congruent* $\triangle$s *by drawing the diagonal* AC.

Proof. Draw the diagonal AC. Then

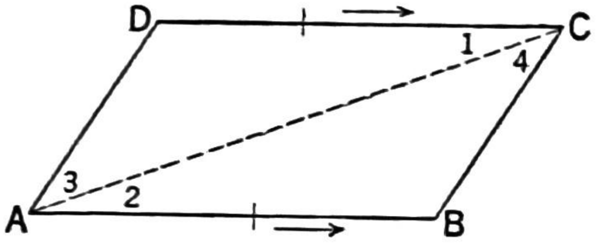
(1) $AD = BC$	(1) Given.
(2) $AB = DC$	(2) Given.
(3) $AC = AC$	(3) Iden.
(4) $\therefore \triangle ABC \cong \triangle ADC$	(4) s. s. s. = s. s. s.
(5) $\therefore \angle 1 = \angle 2$ and $\angle 3 = \angle 4$	(5) Corr. parts of congruent $\triangle$s are equal.
(6) $\therefore AB \parallel DC$ and $AD \parallel BC$	(6) 51.
(7) $\therefore ABCD$ is a ▱.	(7) 81.

Q.E.D.

$ABCD$ is a parallelogram. AD is prolonged to E, CB is prolonged to F, and $DE = BF$. If CE and AF are drawn, prove that $AFCE$ is a parallelogram.

Proposition XVII. Theorem

89. If two sides of a quadrilateral are equal and parallel, the figure is a parallelogram.



Given quad. $ABCD$, with $AB = DC$ and $AB \parallel DC$.

Prove $ABCD$ is a parallelogram.

Analysis. The burden of proof is to show that $AD \parallel BC$. ∴ *we must try to get a pair of alt. int.* ∠s *equal. This we can do by making them corr. parts of congruent* △s.

Proof. Draw the diagonal AC. Then

(1) $AB = DC$	(1) Given.
(2) $\angle 1 = \angle 2$	(2) 56.
(3) $AC = AC$	(3) Iden.
(4) ∴ $\triangle ABC \cong \triangle ADC$	(4) s. a. s. = s. a. s.
(5) ∴ $\angle 3 = \angle 4$	(5) Corr. parts of congruent △s are equal.
(6) ∴ $AD \parallel BC$	(6) 51.
(7) ∴ $ABCD$ is a ▱.	(7) 81.

Q.E.D.

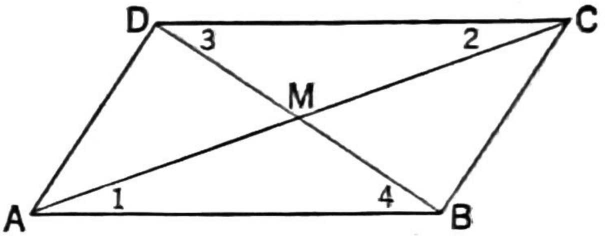
Complete each of the following statements:

(*a*) The diagonals of an equilateral parallelogram are . . . to each other.

(*b*) If the diagonals of a parallelogram are equal, the figure is a. . . .

Proposition XVIII. Theorem

90. The diagonals of a parallelogram bisect each other.



Given ▱ $ABCD$, with its diagonals AC and DB cutting each other at M.

Prove $AM = MC$ and $DM = MB$.

Plan is to get congruent triangles.

Proof.

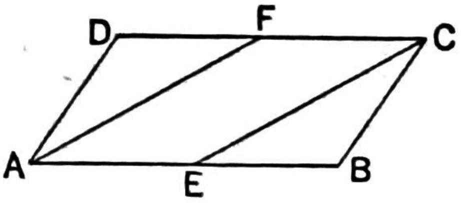
(1) $AB = DC$	(1) 84.
(2) $\angle 1 = \angle 2$	(2) 56.
(3) $\angle 3 = \angle 4$	(3) 56.
(4) $\therefore \triangle AMB \cong \triangle DMC$	(4) a. s. a. = a. s. a.
(5) $\therefore AM = MC$ and $DM = MB$	(5) Corr. parts of congruent $\triangle$s are equal.

Q.E.D.

Ex. 73. Conversely, if the diagonals of a quadrilateral bisect each other, the figure is a parallelogram.

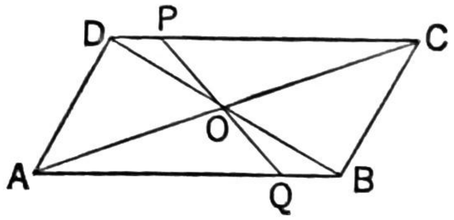
Sug. Aim to establish 88 or 89.

Exercises

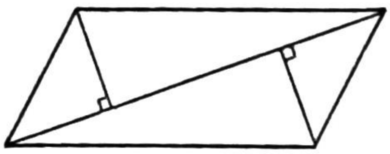


Ex. 74. $ABCD$ is a parallelogram with E and F the midpoints of AB and CD respectively. Prove that $AECF$ is a parallelogram.

Ex. 75. Any line through the point of intersection of the diagonals of a parallelogram and terminated by the sides is bisected at that point.

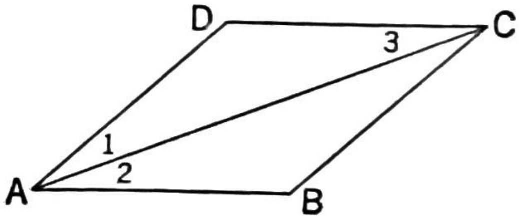


Ex. 76. The perpendiculars to a diagonal of a parallelogram from the opposite vertices are equal.



Ex. 77. If a diagonal bisects an angle of a parallelogram, the figure is equilateral.

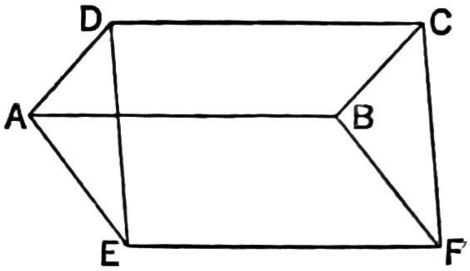
Outline of proof. $\angle 1 = \angle 2 = \angle 3$, $\therefore AD = DC$; but $AD = BC$ and $DC = AB$, $\therefore AB = BC = DC = AD$



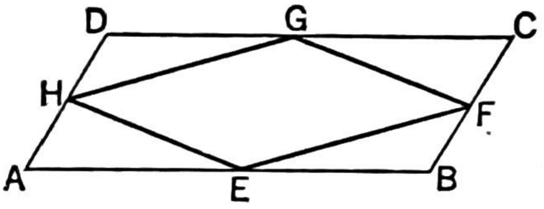
91. *Def.* A parallelogram that has all its sides equal, and all its angles oblique, is called a *rhombus*. Thus in Ex. 77 we are really asked to prove *ABCD* a rhombus.

Ex. 78. *ABCD* is a ▱ and *ABFE* is a ▱. Prove *DCFE* a ▱.

Sug. Prove $DC \parallel EF$, also $DC = EF$

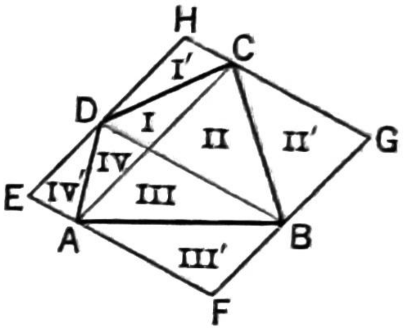


Ex. 79. If the midpoints of the adjacent sides of a parallelogram are joined in order, another parallelogram is formed.



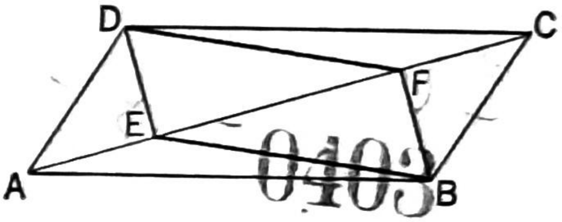
Sug. Get pairs of congruent △s. Then $EF = HG$ and $HE = GF$. $\therefore$ *EFGH* is a ▱.

Ex. 80. Lines through the vertices of any quadrilateral parallel to the diagonals form a parallelogram equal to twice the quadrilateral.



Sug. First prove *EFGH* a ▱. Then $I = I'$, etc. (a diag. divides a ▱ into two congruent ⚠). ∴ $EFGH = 2\,I + 2\,II + 2\,III + 2\,IV$, etc.

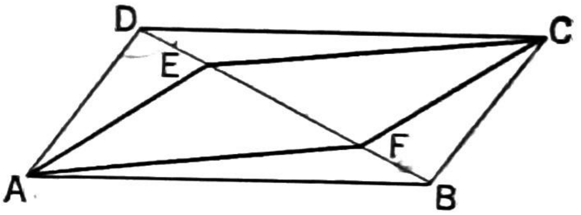
Ex. 81. *ABCD* is a ▱, $DE \perp AC$, $BF \perp AC$. Prove *DEBF* is a ▱.



0403

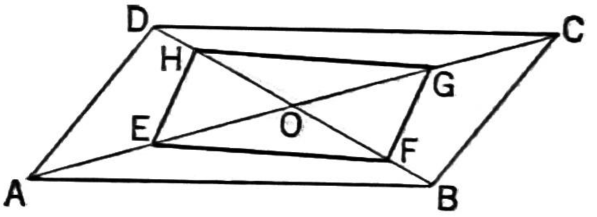
Sug. Prove $DE = FB$, also $DE \parallel FB$.

Ex. 82. *ABCD* is a ▱, *DB* a diagonal, $DE = FB$. Prove *AFCE* is a ▱.



Sug. Prove $EC =$ and $\parallel AF$.

Ex. 83. *ABCD* is a ▱, *AC* and *BD* its diagonals. $DH = FB$, $AE = GC$. Prove *EFGH* is a ▱.



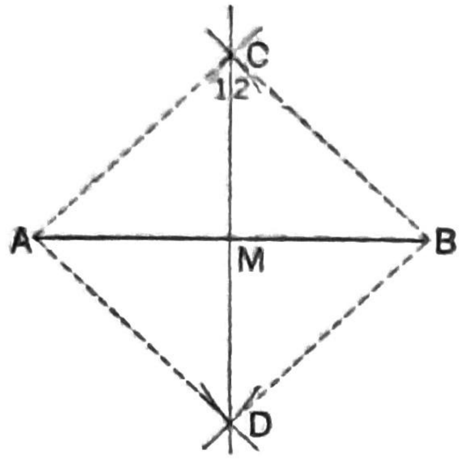
Sug. Prove $OH = OF$, and $OE = OG$. Then use ex. 73.

FUNDAMENTAL CONSTRUCTIONS

92. The only instruments we are permitted to use for the constructions of geometry are an unmarked ruler (a straightedge) and a pair of compasses. A geometric construction that is required to be made is called a problem. There are four distinct steps in solving a problem: (1) State what is given; (2) State what is required to be constructed; (3) Tell definitely how to do it (called the construction); (4) Prove that the construction is correct. Often there is a fifth step called the discussion, where we investigate under what conditions there is more than one solution or no solution at all. With the aid of the fundamental constructions, more difficult and complicated problems can be solved.

Proposition XIX. Problem

93. To bisect a given line.



Given the line AB.

Required to bisect AB.

Analysis. If the problem were done, then the two parts would be equal. We can get two lines equal if we can make them corr. parts of congruent $\triangle$s. $\therefore$ *we construct two congruent* $\triangle$s.

Construction. For details of procedure see 25, problem 2.

Proof. Draw AC, BC, AD, BD. Then

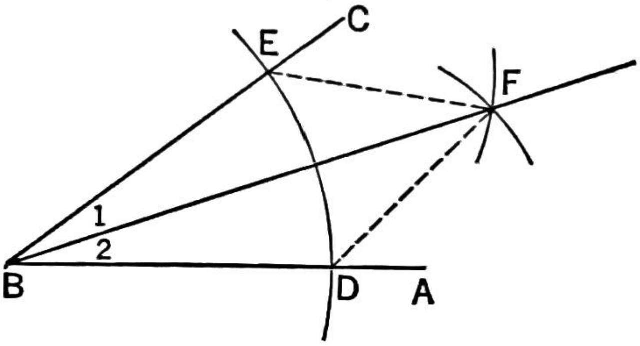
(1) $AC = BC$	(1) Const.
(2) $AD = BD$	(2) Const.
(3) $CD = CD$	(3) Iden.
(4) $\therefore \triangle ACD \cong \triangle BCD$	(4) s. s. s. = s. s. s.
(5) $\therefore \angle 1 = \angle 2$	(5) Corr. parts of congruent $\triangle$s are equal.
(6) $\therefore \triangle ACM \cong \triangle BCM$	(6) s. a. s. = s. a. s.
(7) $\therefore AM = MB$	(7) Corr. parts of congruent $\triangle$s are equal.

Q.E.D.

Note: We could have proved the above problem more briefly by using the corollary to Prop. XIII. Thus, C is equidistant from A and B, D is equidistant from A and B, $\therefore CD$ is $\perp$ bisector of AB.

Proposition XX. Problem

94. To bisect a given angle.



Given $\angle ABC$.

Required to bisect $\angle ABC$.

Plan is to make the two parts corresponding parts of congruent $\triangle$s.

Construction. For details of procedure, see 25, problem 1.

Proof. Draw EF and DF. Then

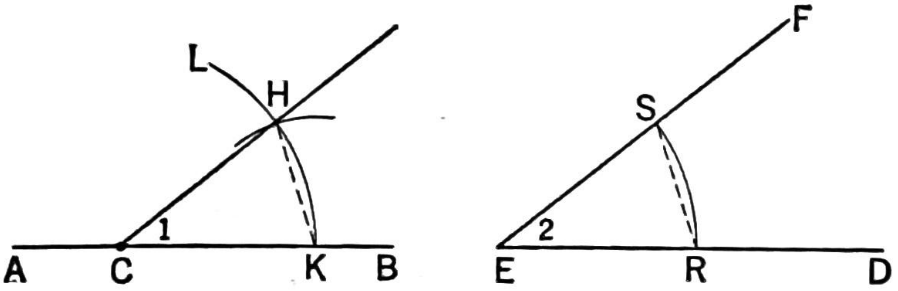
(1) $BE = BD$	(1) Const.
(2) $EF = DF$	(2) Const.
(3) $BF = BF$	(3) Iden.
(4) $\therefore \triangle BEF \cong \triangle BDF$	(4) s. s. s. = s. s. s.
(5) $\therefore \angle 1 = \angle 2$	(5) Corr. parts of congruent $\triangle$s are equal.

Q.E.D.

Query: Can you divide an angle into four equal parts? eight equal parts?

Proposition XXI. Problem

95. From a given point on a given line, to construct a line which shall make with the given line an angle equal to a given angle.



Given line AB, C a point in AB, and $\angle DEF$.

Required to draw a line from C which will make with AB an angle equal to $\angle DEF$.

Plan is to get the angles to be corresponding parts of congruent $\triangle$s.

Construction. For details of procedure, see 25, problem 3.

Proof. Draw KH and RS. Then

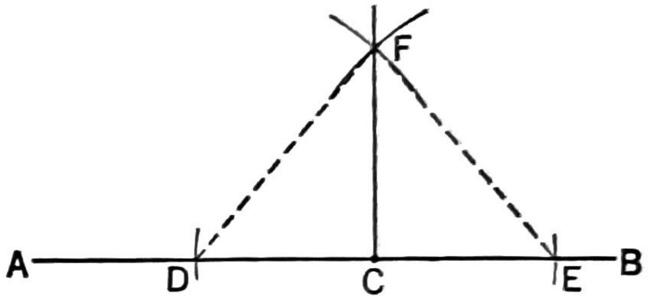
(1) $CH = ES$	(1) Const.
(2) $CK = ER$	(2) Const.
(3) $HK = SR$	(3) Const.
(4) $\therefore \triangle CKH \cong \triangle ERS$	(4) s. s. s. = s. s. s.
(5) $\therefore \angle 1 = \angle 2$	(5) Corr. parts of congruent $\triangle$s are equal.

Q.E.D.

Query: Can you construct an angle twice as large as a given angle? two and one half times as large?

Proposition XXII. Problem

96. At a given point in a given straight line, to draw a perpendicular to that line.



Given point C in the line AB.

Required to draw a perpendicular to AB at C.

Plan is to get two points each equidistant from two other points.

Construction. (*a*) With C as center, and any radius such as CD, make $CE = CD$.

(*b*) With D as center and any radius greater than CD, describe an arc.

(*c*) With E as center and the same radius used in step (*b*) describe an arc cutting the last arc in F.

(*d*) Draw the line through C and F. Then CF is the required perpendicular.

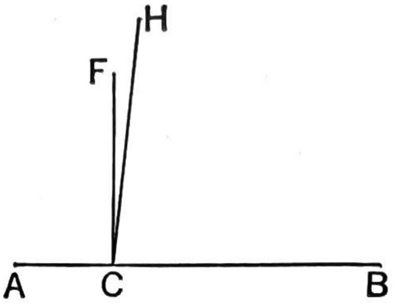
Proof.

(1) $CD = CE$	(1) Const.
(2) $FD = FE$	(2) Const.
(3) $\therefore CF$ is $\perp$ bisector of DE i.e. $CF \perp AB$	(3) 78. Q.E.D.

Query: How would you make the above construction if the point C were one end of the given line?

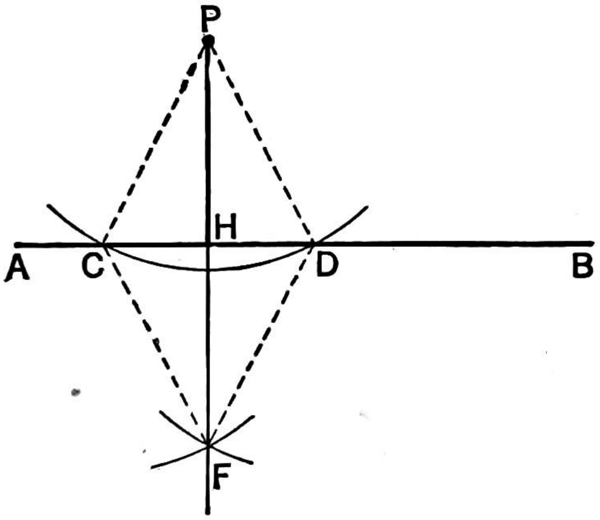
97. *Cor.* At a given point in a given line, one line and only one can be drawn ⊥ that line.

Proof. We have just shown that it is possible to draw one ⊥. If another could be drawn, then we would have part of an angle equal to the whole angle. Thus if CH could also be ⊥ AB at C, then ∠ BCH would be a right angle. But ∠ BCF is a right angle. ∴ ∠ BCH = ∠ BCF (all right angles are equal), which is absurd, since ∠ BCH is only a part of ∠ BCF.



Proposition XXIII. Problem

98. To draw a perpendicular to a given line from a given point outside the line.



Given line AB and any point P not in AB.

Required to draw a perpendicular from P to AB.

Plan is to get two points each equidistant from two other points.

Construction.

(a) With P as center, and a radius sufficiently long, draw an arc cutting AB at C and D.

(*b*) With C as center, and any radius greater than half of CD, draw an arc below AB.

(*c*) With D as center and the same radius used in (*b*), draw an arc cutting the last arc in F.

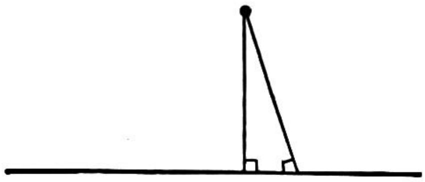
(*d*) Draw the line through P and F, and let it cross AB in H.

Then PH is the required perpendicular.

Proof.

(1) $PC = PD$	(1) Const.
(2) $FC = FD$	(2) Const.
(3) $\therefore PF$ is $\perp$ bisector of CD i.e. $PH \perp AB$	(3) 78.
	Q.E.D.

99. *Cor.* From a given point outside a given line, one line and only one can be drawn $\perp$ that line.



Proof. We have just shown that it is possible to draw one $\perp$. If another could be drawn, then we would have two right angles in the same triangle, which is absurd. (Prop. X, Cor. 2.)

Ex. 84. Construct an angle of 45°.

Ex. 85. Construct the bisectors of the three angles of a triangle.

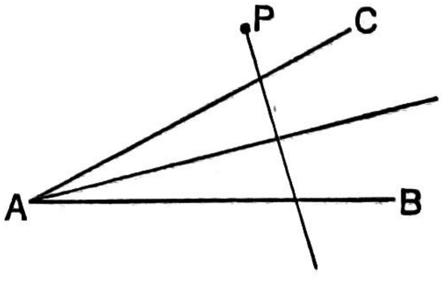
Ex. 86. Construct a square with a given line as a side.

Ex. 87. Construct the three altitudes of an acute triangle.

Ex. 88. Construct the three altitudes of an obtuse triangle.

Ex. 89. Construct the three medians of a triangle.

Ex. 90. Through a given point draw a line which shall make equal angles with the sides of a given angle.

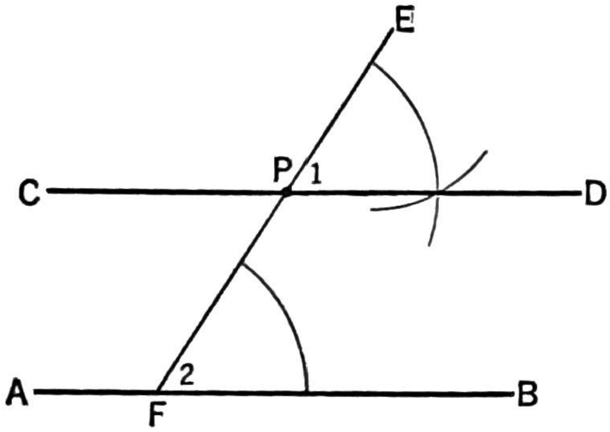


Solution: Bisect the given angle, and through the given point draw a line $\perp$ this bisector.

Ex. 91. Construct the perpendicular bisectors of the three sides of a triangle. Consider also the case of an obtuse triangle.

Proposition XXIV. Problem

100. Through a given point to draw a parallel to a given line.



Given line AB and point P.

Required to draw a line through $P \parallel AB$.

Plan is to make a pair of corresponding $\measuredangle$s equal.

Construction. (*a*) Through P draw any line EF, cutting AB in F.

(*b*) At P construct $\angle 1$ equal to $\angle 2$. (Prop. XXI.) Then CD is the required line.

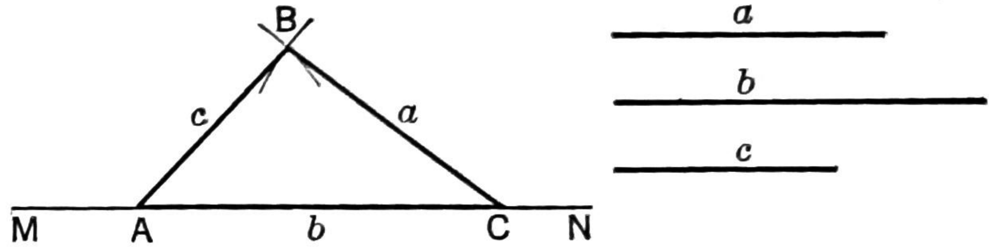
Proof. (1) $\angle 1 = \angle 2$ | (1) Const.
(2) $\therefore CD \parallel AB$ | (2) 53.

Q.E.D.

Query: If through each vertex of a triangle a line is drawn parallel to the opposite side, what kind of a figure will be formed? How will it compare with the original triangle?

Proposition XXV. Problem

101. To construct a triangle when the three sides are given.



Given lines *a*, *b*, and *c*.

Required to draw a triangle with sides equal to *a*, *b*, *c*.

Construction. (*a*) Draw a line MN and on it mark off AC equal to *b*.

(*b*) With A as center and *c* as radius draw an arc.

(*c*) With C as center and *a* as radius draw an arc cutting the last arc at B.

(*d*) Draw AB and CB. Then $\triangle ABC$ is the required triangle.

Proof. $AC = b$, $AB = c$, $CB = a$ by const.

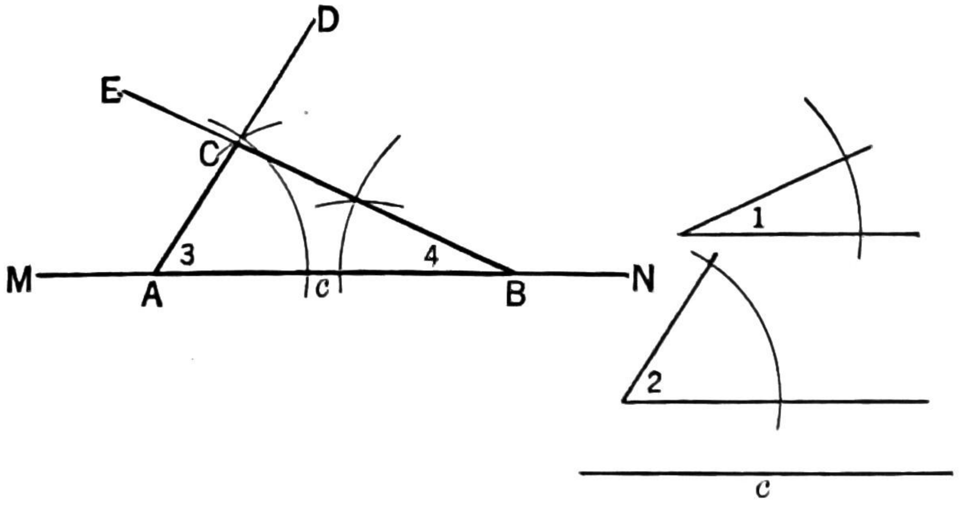
Discussion. In order that it shall be possible to construct a triangle with lines *a*, *b*, *c* as sides, the lines must be of such length that the sum of *any* two of them is greater than the third.

102. It is convenient to label the side opposite $\angle A$ as *a*, that opposite $\angle B$ as *b*, that opposite $\angle C$ as *c*.

Query: Can you construct a quadrilateral when the four sides and one diagonal are given?

Proposition XXVI. Problem

103. To construct a triangle when two angles and the included side are given.



Given line c and ∠s 1 and 2.

Required to construct a △ in which ∠s 1 and 2 are two angles and c the included side.

Construction. (*a*) Draw a line MN and on it mark off AB equal to c.

(*b*) At A construct ∠ 3 equal to ∠ 2.
(*c*) At B construct ∠ 4 equal to ∠ 1.
(*d*) Note that AD and BE intersect at C.

Then △ ABC is the required triangle.

Proof. ∠ 3 = ∠ 2, ∠ 4 = ∠ 1, and the included side $AB = c$ by const.

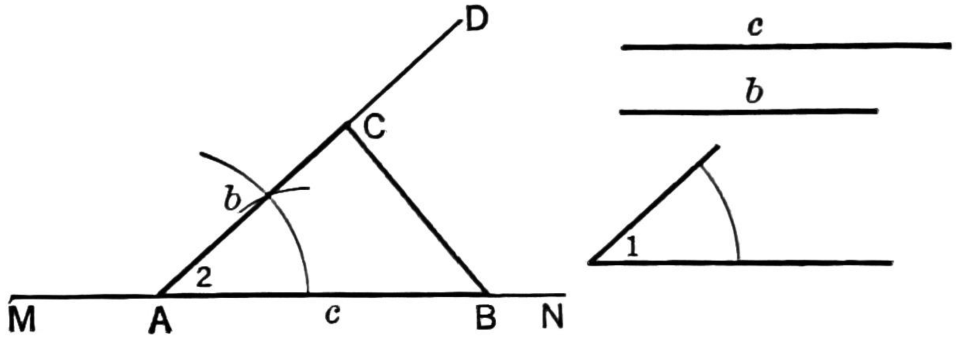
Discussion. The problem is impossible if the sum of the two given angles is equal to or greater than a straight angle.

104. *Cor.* To construct a triangle when two angles and the side opposite one of them are given.

Sug. Find the third angle, and then the problem becomes Prop. XXVI. The third angle can be found by noting that the sum of the angles about a point, on the same side of a straight line passing through that point, is a straight angle.

Proposition XXVII. Problem

105. To construct a triangle when two sides and the included angle are given.



Given b, c and $\angle 1$.

Required to construct a triangle in which b and c are two sides and $\angle 1$ is the included $\angle$.

Construction. (*a*) Draw a line MN, and on it mark off AB equal to c.

(*b*) At A construct $\angle 2$ equal to $\angle 1$.

(*c*) On AD mark off AC equal to b.

(*d*) Draw BC. Then $\triangle ABC$ is the required $\triangle$.

Proof. $AB = c$, $AC = b$, $\angle 2 = \angle 1$ by const.

Exercises

Ex. 92. Construct a right triangle, having given the hypotenuse and one leg.

Ex. 93. Construct an angle of 60°.

Ex. 94. Construct an equilateral triangle, having given the altitude.

Ex. 95. Construct an isosceles triangle, having given the altitude to the base and one of the legs.

Ex. 96. Construct an isosceles triangle, having given the angle at the vertex and the altitude to the base.

Ex. 97. Construct a parallelogram when two sides and the included angle are given.

Ex. 98. Construct a square having given the diagonal.

106. *Def.* A *square* is a parallelogram that has all its sides equal and all its angles right angles.

Ex. 99. Construct a right triangle when a leg and the opposite angle are given.

107. *Def.* A *segment* of a line is any portion of that line. Thus AC is a segment of AB.

GROUP III

Complete each of the following statements.

(*a*) The converse of the theorem, "The base angles of an isosceles triangle are equal," is . . . .

(*b*) In $\triangle ABC$, $AC = BC$ and BD bisects $\angle B$. If $\angle BDC = 96°$, then $\angle C$ equals . . . degrees.

(*c*) $ABCD$ is a parallelogram and parallel lines are drawn from D and B meeting diagonal AC at E and F respectively. Then the figure $DBEF$ is a . . . .

(*d*) State whether it is possible or impossible to construct a triangle with sides $3\frac{1}{2}''$, $4\frac{1}{2}''$, and $8''$.

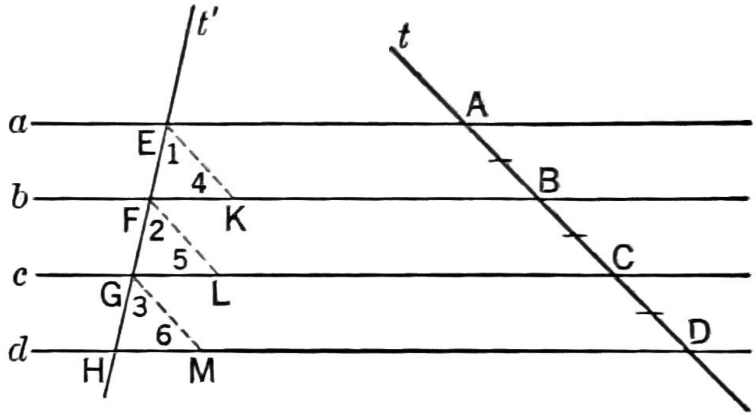
(*e*) AB and CD are parallels cut by transversal EF at G and H respectively. $\angle EGB$ is 48°. Hence $\angle FHC$ is . . . degrees.

(*f*) If the vertex angle of an isosceles triangle is 40°, the angle formed by the bisectors of the base angles is . . . degrees.

SEGMENTS OF TRANSVERSALS

Proposition XXVIII. Theorem

108. If three or more parallels cut off equal segments on one transversal, they cut off equal segments on every transversal.



Given the parallels a, b, c, d cutting off the equal segments AB, BC, CD on the transversal t, and that t' is another transversal of these parallels.

Prove $EF = FG = GH$.

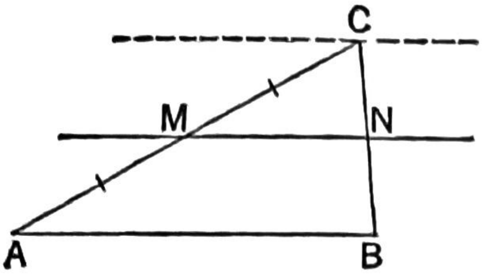
Plan is to get congruent $\triangle$ *of which* EF, FG, GH *shall be corresponding parts. By drawing* $\|$*s to* t *we shall have the* $\triangle$ *congruent by a. s. a.*

Proof. Through E, F, and G draw the lines EK, FL and $GM \parallel t$. Then

(1) $EK \parallel FL \parallel GM$	(1) 61.
(2) $EK = AB$, $FL = BC$, $GM = CD$	(2) 84.
(3) but $AB = BC = CD$	(3) Given.
(4) $\therefore EK = FL = GM$	(4) Ax. 1.
(5) $\angle 1 = \angle 2 = \angle 3$	(5) 58.
(6) $\angle 4 = \angle 5 = \angle 6$	(6) 64.
(7) $\therefore \triangle EFK \cong \triangle FGL \cong \triangle GHM$	(7) a. s. a. = a. s. a.
(8) $\therefore EF = FG = GH$	(8) Why?

Q.E.D.

109. *Cor.* A parallel to one side of a triangle, bisecting a second side, bisects the third side also.



GROUP IV

Choose the correct word or words in each parenthesis.

(*a*) "Only one straight line can be drawn connecting two given points" is a (definition, postulate, theorem).

(*b*) If two parallel lines are cut by a transversal, the interior angles on the same side of the transversal are (equal, complementary, adjacent, supplementary).

(*c*) If two adjacent angles have their exterior sides in the same straight line they are (right, equal, supplementary, complementary).

(*d*) Two right triangles are congruent if the respectively equal parts are an arm and the (right angle, hypotenuse, acute angle).

(*e*) If the opposite angles of a quadrilateral are equal, the figure is a (square, rhombus, parallelogram).

(*f*) A quadrilateral is a rhombus if its (sides, diagonals, opposite angles) are equal.

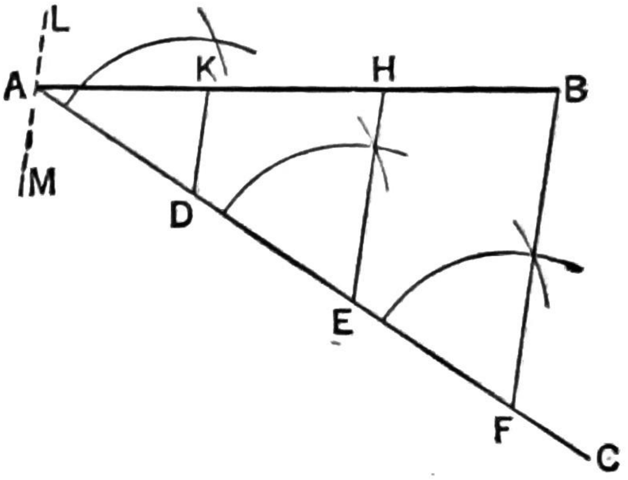
(*g*) Four parallels cut off three equal segments on a transversal such that each segment is 2″. Hence the segments cut off by these parallels on any other transversal must each be (two inches, more than two inches, cannot tell).

(*h*) $ABCD$ is a parallelogram. AE, the bisector of $\angle A$, meets CD at E. CF, the bisector of $\angle C$, meets AB at F. The figure $AFCE$ is a (rhombus, parallelogram, rectangle).

(*i*) The side AC of $\triangle ABC$ is prolonged to D. The bisector of $\angle ACB$ meets AB at R, and the bisector of $\angle BCD$ meets AB prolonged at S. The $\triangle CRS$ is (right, obtuse, cannot tell).

Proposition XXIX. Problem

110. To divide a given line into any number of equal parts.



Given line AB.

Required to divide AB into any number of equal parts (say three).

Plan is to get ||*s cutting off equal segments on one transversal. Then they must cut off equal segments on* AB.

Construction. (*a*) Draw AC making any angle with AB.

(*b*) On AC, starting at A and using any length, mark off that length as many times as is indicated by the number of parts into which AB is to be divided.

(*c*) Connect the last point F with B.

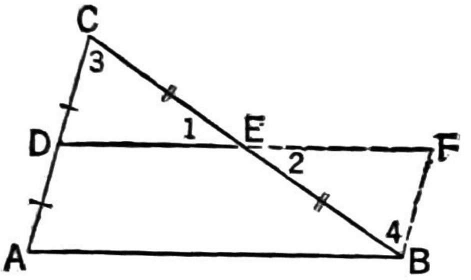
(*d*) Through E and D draw ||s to FB, meeting AB at H and K. The line AB is thus divided into three equal parts.

Proof. Through A draw $LM \parallel BF$. Then

(1) LM, KD, HE, and BF are || each other	(1) 61.
(2) $AD = DE = EF$	(2) Const.
(3) $\therefore AK = KH = HB$	(3) 108.
	Q.E.D.

Proposition XXX. Theorem

111. The line which joins the midpoints of two sides of a triangle is parallel to the third side and is equal to half the third side.



Given $\triangle ABC$, with D the midpoint of AC, and E the midpoint of BC.

Prove $DE \parallel AB$, and $DE = \frac{1}{2} AB$.

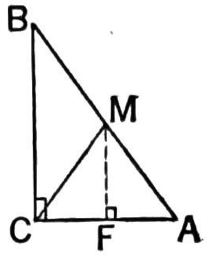
Analysis. Since we are to prove $AB = 2\,DE$, *that suggests that we prolong* DE *its own length to* F. *Now if we can prove* $DF = AB$ *and* $DF \parallel AB$ *the proposition will be proved. Hence we must aim to prove* $ADFB$ *a* ▱.

Proof. Prolong DE its own length to F. Draw BF.

(1) $CE = EB$	(1) Given.
(2) $DE = EF$	(2) Const.
(3) $\angle 1 = \angle 2$	(3) 28.
(4) $\therefore \triangle CDE \cong \triangle EBF$	(4) s. a. s. = s. a. s.
(5) $\therefore \angle 3 = \angle 4$, $DC = BF$	(5) Corr. parts of congruent △s are equal.
(6) $\therefore AC \parallel BF$	(6) 51.
(7) $AD = DC$	(7) Given.
(8) $\therefore AD = BF$	(8) Ax. 1.
(9) $\therefore ADFB$ is a ▱	(9) 89.
(10) $\therefore DE \parallel AB$	(10) 81.
(11) and $DF = AB$	(11) 84.
(12) but $DE = \frac{1}{2} DF$	(12) Const.
(13) $\therefore DE = \frac{1}{2} AB$	(13) Sub.

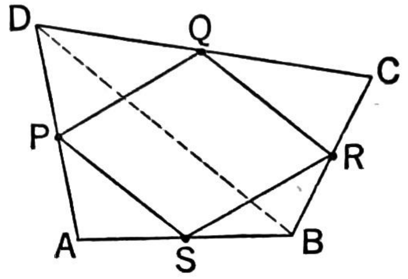
Q.E.D.

Exercises



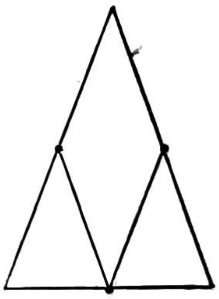
Ex. 100. The median to the hypotenuse of a right triangle is one half the hypotenuse.

Sug. Draw $MF \perp CA$.



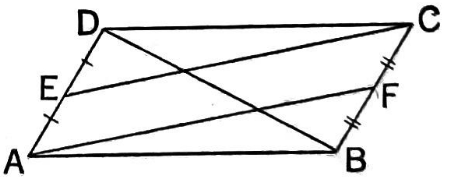
Ex. 101. The lines joining the midpoints of the sides of any quadrilateral, taken in order, form a parallelogram.

Sug. Draw DB, and thus prove $QR = PS$, and $QR \parallel PS$.



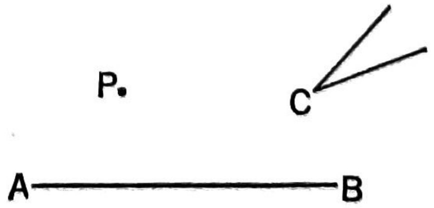
Ex. 102. If the midpoints of the legs of an isosceles triangle are joined to the midpoint of the base, a rhombus is formed.

Ex. 103. Given three fixed points A, B, and C, not in a straight line. Construct a triangle the midpoints of whose sides shall be A, B, and C.



Ex. 104. Lines joining the midpoints of two opposite sides of a parallelogram to the ends of a diagonal trisect (cut into three equal parts) the other diagonal.

Sug. First prove $AF \parallel EC$.



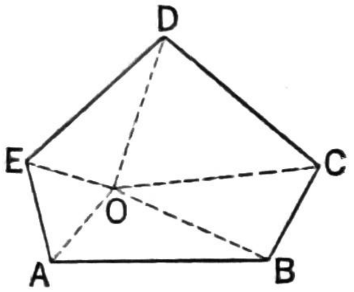
Ex. 105. Through a given point outside a given line, draw a line which shall make with the given line an angle equal to a given angle.

Ex. 106. Construct an equilateral triangle whose perimeter shall equal a given line. (The perimeter is the total length of the sides.)

ANGLES OF A POLYGON

Proposition XXXI. Theorem

112. **The sum of all the angles of a polygon of n sides is equal to $(n - 2)$ straight angles.**



Given $ABC\ldots$, a polygon of n sides (where n stands for any whole number not less than 3).

Prove $\angle A + \angle B + \angle C +$ etc. $= (n - 2)$ straight angles.

Plan is to divide the polygon into triangles.

Proof. Take any point O within the polygon, and connect it with the vertices, thus forming as many △s as there are sides (since there is a △ for every side). Then

(1) The sum of the angles of each triangle = one straight angle	(1) 66.
(2) ∴ the sum of the angles of all the △s $= n$ straight angles	(2) Since there are n △s
(3) But the sum of all the angles about O = 2 straight angles	(3) Ex. 21.
(4) ∴ the sum of the angles of all the △s − sum of all the angles about $O = (n - 2)$ straight angles	(4) Ax. 3.
(5) ∴ $\angle A + \angle B + \angle C +$ etc. $= (n - 2)$ straight angles	(5) Sub.

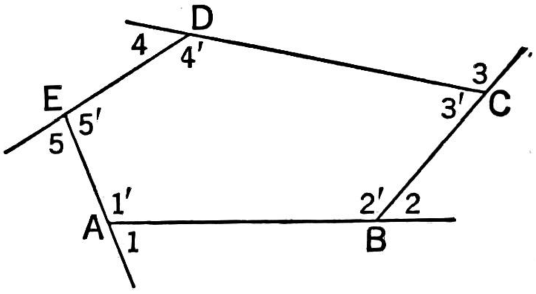
Q.E.D.

Thus if a polygon has 12 sides, the sum of all the angles = (12 − 2) straight angles = 10(180°) = 1800°.

113. *Cor.* 1. In an equiangular polygon of n sides, each angle equals $\frac{n-2}{n}$ straight angles.

(because each ∠ will be $\frac{1}{n}$ of the total sum). E.g. each ∠ of an equiangular polygon of 5 sides is $\frac{1}{5}$ of (5 − 2) straight ∠s, or $\frac{1}{5}$ of 3 straight ∠s, or $\frac{1}{5}$ of 540°, or 108°.

114. *Cor.* 2. The sum of the exterior angles of a polygon, made by producing the sides in succession, is equal to two straight angles.

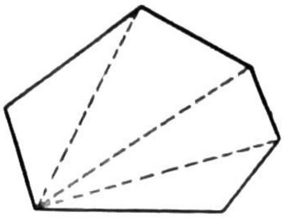


Outline of proof. ∠ 1 + ∠ 1′ = one straight ∠.
∠ 2 + ∠ 2′ = one straight ∠.
and so on at every vertex.

∴ sum of all the interior ∠s + sum of all the exterior ∠s = n straight ∠s
but sum of all the interior ∠s = (n − 2) straight ∠s
(subtracting) ∴ sum of all the exterior ∠s = 2 straight ∠s

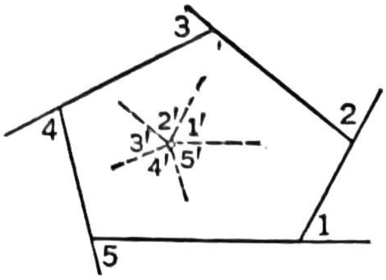
115. Note the interesting fact that the sum of the exterior angles formed by producing the sides in succession is always the same, no matter how many sides the polygon has; but the sum of the interior angles varies with the number of sides. In other words, the sum in the former case does not depend on the value of n, whereas in the latter case it does.

Exercises



Ex. 107. Prove Prop. XXXI by taking the point O at one of the vertices of the polygon.

Sug. $(n - 2)$ triangles are thus formed.



Ex. 108. Prove Prop. XXXI, Cor. 2, by transferring the exterior angles about any point O.

Sug. Through O draw ‖s to the sides.

Ex. 109. Fill in the following table.

Kind of polygon	*No. of degrees in the sum of all the angles*	*No. of degrees in each angle, assuming the polygon equiangular*
polygon of 3 sides, or triangle		
" " 4 sides, or quadrilateral		
" " 5 sides, or pentagon		
" " 6 sides, or hexagon		
" " 8 sides, or octagon		
" " 10 sides, or decagon		
" " n sides, or n-gon		

Ex. 110. How many sides has a polygon the sum of whose angles is 900°?

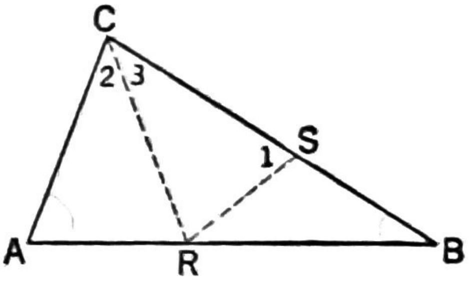
Ex. 111. How many sides has an equiangular polygon, four of whose angles are together equal to seven right angles?

Ex. 112. How many sides has a polygon the sum of whose interior angles equals six times the sum of its exterior angles? (The exterior angles here referred to are formed by producing the sides in succession.)

INEQUALITIES

Proposition XXXII. Theorem

116. If one side of a triangle is greater than a second side, the angle opposite the first side is greater than the angle opposite the second side.



Given $\triangle ABC$ with $BC > AC$.

Prove $\angle A > \angle B$.

Analysis. If we can get $\angle A$ *equal to an angle which is known to be greater than* $\angle B$, *then the theorem is proved. Since* $BC > AC$, *we can lay off on* BC *a distance* CS *equal to* AC, *and we are sure that* S *must fall somewhere between* C *and* B. *By constructing congruent* $\triangle$ *we can get* $\angle A = \angle 1$, *and since* $\angle 1 > \angle B$, *then* $\angle A > \angle B$.

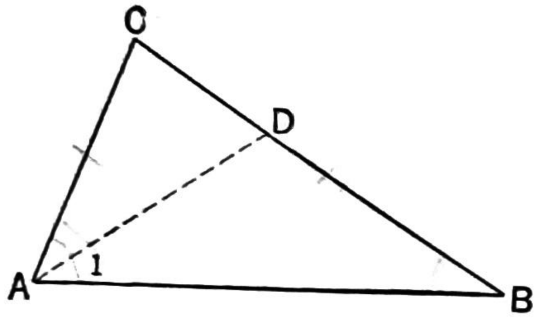
Proof. On CB lay off $CS = CA$. Bisect $\angle C$ by the line CR, and draw RS. Then

(1) $CA = CS$	(1) Const.
(2) $\angle 2 = \angle 3$	(2) Const.
(3) $CR = CR$	(3) Iden.
(4) $\therefore \triangle ACR \cong \triangle RCS$	(4) s. a. s. = s. a. s.
(5) $\therefore \angle A = \angle 1$	(5) Corr. parts of congruent $\triangle$ are equal.
(6) but $\angle 1 > \angle B$	(6) 49.
(7) $\therefore \angle A > \angle B$	(7) Sub.

Q.E.D.

Proposition XXXIII. Theorem

117. If one angle of a triangle is greater than a second angle, the side opposite the first angle is greater than the side opposite the second angle.



Given $\triangle ABC$ with $\angle A > \angle B$.

Prove $BC > AC$.

Analysis. Since $\angle A > \angle B$ *we can mark off* $\angle B$ *as part of* $\angle A$. *We will thus get* BC *equal to the broken line* CDA. *But* $CDA > AC$. $\therefore BC > AC$.

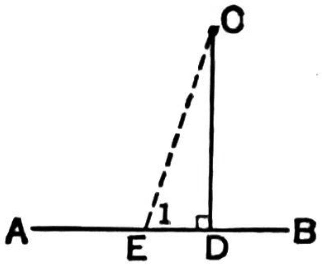
Proof. Draw AD so as to make $\angle 1 = \angle B$. Then

(1) $AD = BD$	(1) 73.
(2) but $AD + DC > AC$	(2) A straight line is the shortest distance between two points.
$\therefore BD + DC > AC$	(3) Sub.
$\therefore BC > AC$	(4) Sub.

Q.E.D.

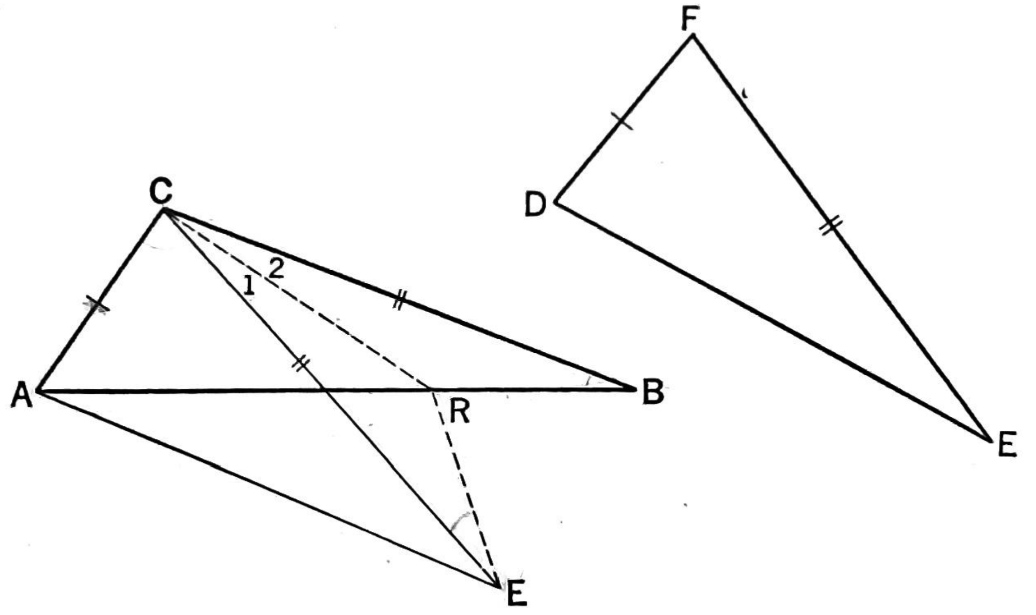
118. *Cor.* The perpendicular is the shortest line that can be drawn from a given point to a given line.

Outline of proof. CE is any other line from C to AB. Since $\angle D > \angle 1$, it follows that $CE > CD$.



Proposition XXXIV. Theorem

119. If two triangles have two sides of one equal respectively to two sides of the other and the included angles unequal, the triangle which has the greater included angle has the greater third side.



Given $\triangle$ ABC and DEF with $AC = DF$, $BC = EF$, $\angle ACB > \angle F$.

Prove $AB > DE$.

Analysis. One way to compare the size of things is to bring them close together. $\therefore$ *place* $\triangle DEF$ *in the position* AEC. *The problem then is to prove* $AB > AE$. *This we can do, if we can prove* AB *equal to a broken line between* A *and* E. *We must therefore find a point* R *on* AB *such that* $RE = RB$. *This would make* $\triangle CRE \cong \triangle CRB$ *by s. s. s.* $\therefore \angle 1 = \angle 2$, *suggesting that we bisect* $\angle ECB$.

Plan is to prove $AB = ARE$ *which* $> AE$.

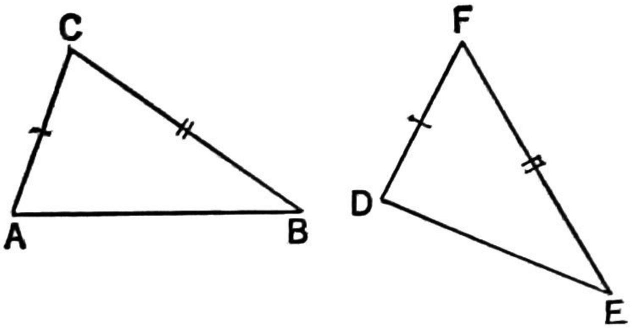
Proof. Place $\triangle DEF$ upon $\triangle ABC$ so that DF shall fall upon its equal AC, and the vertex E to the right of AC. (Why must FE fall within $\angle ACB$?) Bisect $\angle ECB$ by the line CR. Draw ER. Then

(1) $AR + RE > AE$	(1) A straight line is the shortest distance between two points.
(2) but $RE = RB$	(2) Corr. parts of congruent $\triangle$s are equal. ($\triangle ECR \cong \triangle BCR$ by s. a. s.)
(3) $\therefore AR + RB > AE$	(3) Sub.
(4) $\therefore AB > AE$	(4) Whole equals sum of its parts.
(5) $\therefore AB > DE$	(5) Sub.

Q.E.D.

Proposition XXXV

120. If two triangles have two sides of one equal respectively to two sides of the other and the third sides unequal, the triangle which has the greater third side has the greater angle opposite.



Given $\triangle$s ABC and DEF with $AC = DF$, $BC = EF$, $AB > DE$.

Prove $\angle C > \angle F$.

Analysis. If we can show that it is impossible for ∠ C *to equal* ∠ F *or for* ∠ C *to be less than* ∠ F*, then* ∠ C *must be greater than* ∠ F*.*

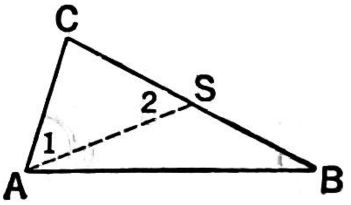
Proof. ∠ C is either equal to ∠ F, less than ∠ F, or greater than ∠ F.

(1) If $\angle C = \angle F$ then $\triangle ABC \cong \triangle DEF$ (by s. a. s. = s. a. s.)
then $AB = DE$ (corr. parts of congruent ▲s are equal)
but $AB > DE$ (given)
∴ the first supposition leads to a contradiction of what is given.

(2) If $\angle C < \angle F$
then $AB < DE$ (Prop. XXXIV)
but $AB > DE$ (Given)
∴ the second supposition leads to a contradiction of what is given. Hence there is only one possibility left, that is $\angle C > \angle F$. Q.E.D.

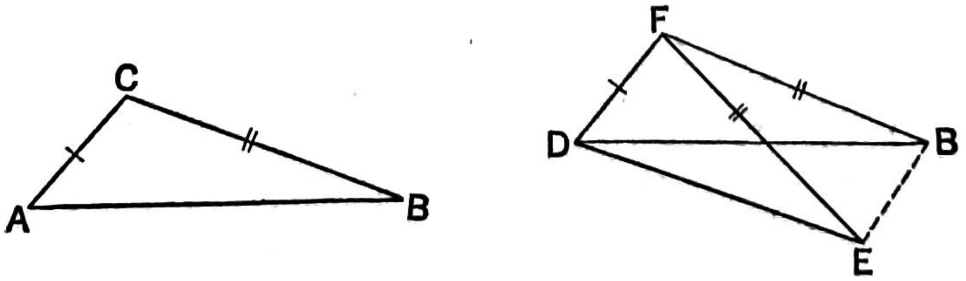
Ex. 113. Prove Prop. XXXII by laying off $CS = CA$, and drawing AS.

Sug. $\angle CAB > \angle 1 > \angle B$.



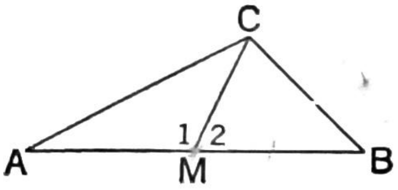
Ex. 114. Prove Prop. XXXIII by the indirect method, i.e. BC is either equal to AC, less than AC, or greater than AC.

Ex. 115. Prove Prop. XXXIV by using the placing and construction as indicated in the annexed figure.

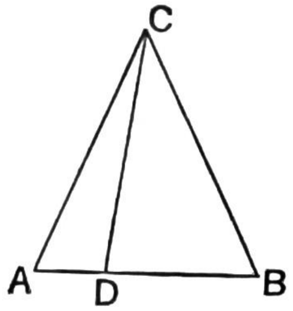


Sug. Try to prove $\angle DEB > \angle DBE$.

Ex. 116. In $\triangle ABC$ it is known that $AC > BC$ and that M is the midpoint of AB. Prove $\angle 1 > \angle 2$.

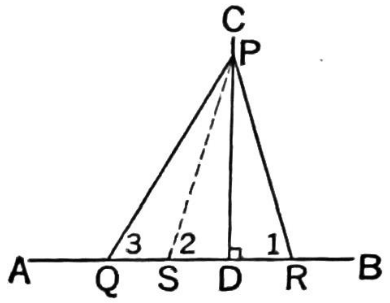


Ex. 117. If D is any point in the base AB of isos. $\triangle ABC$, prove that DC is less than either AC or BC.



Ex. 118. If in $\triangle ABC$, $AB > AC$, and $\angle A = 60°$, which is the greatest $\angle$ of the $\triangle$?

Ex. 119. Of two lines drawn from a point in a perpendicular to a given line, cutting off on the given line unequal segments from the foot of the perpendicular, the more remote is the greater.

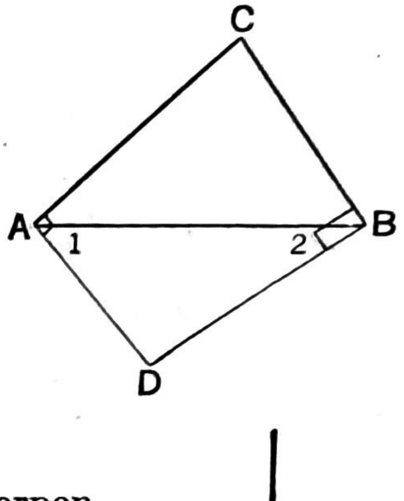


Sug. Lay off $DS = DR$, and draw PS. Try to prove $\angle 1 > \angle 3$.

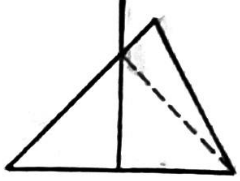
Ex. 120. The sum of the altitudes of a triangle is less than the perimeter of the triangle.

Ex. 121. In the $\triangle ABC$, $AC > BC$. AD is drawn $\perp AC$, $BD \perp BC$. Prove $BD > AD$.

Sug. Try to prove $\angle 1 > \angle 2$.

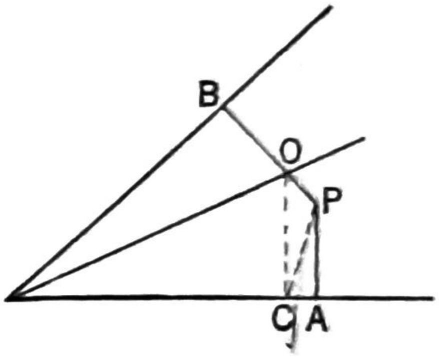


Ex. 122. Any point outside the perpendicular bisector of a line is not equidistant from the ends of the line.

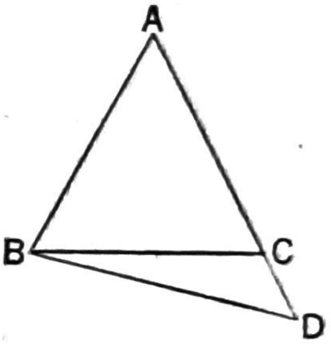


Ex. 123. Any point within an angle not in the bisector of the angle is unequally distant from the sides of the angle.

Sug. $PO + OC > PC > PA$.

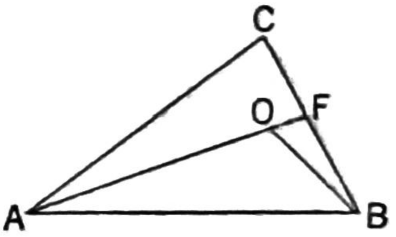


Ex. 124. If AC, a side of an equilateral $\triangle ABC$, is produced to D, and BD is drawn, then $AD > BD > AB$.



Ex. 125. If from any point O within a $\triangle ABC$ OA and OB are drawn, then $CA + CB > OA + OB$.

Outline of proof. Prolong AO to F. Then $AC + CF > AO + OF$, and $OF + FB > OB$; add these inequalities and simplify.



Ex. 126. The sum of the sides of any quadrilateral is greater than the sum of its diagonals.

GROUP V

Complete each of the following statements:

(*a*) The difference between any two sides of a triangle is . . . the third side.

(*b*) The line joining the midpoints of two sides of a $\triangle$ is 6″. Hence the third side is . . . inches.

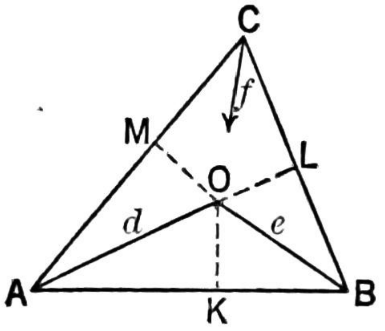
(*c*) Two adjacent sides of a parallelogram are 4″ and 6″. Hence the lengths of the diagonals must be less than . . . inches but greater than . . . inches.

(*d*) If an interior angle of an equiangular polygon is 168°, the polygon has . . . sides.

CONCURRENT LINES

Proposition XXXVI. Theorem

121. The bisectors of the angles of a triangle meet in a point which is equidistant from the sides of the triangle.



Given $\triangle ABC$ with d, e, f the bisectors of $\angle$s A, B, C respectively.

Prove d, e, f meet in a point O which is equidistant from AB, BC, and AC.

Plan is to show that d *and* e *must meet in some point, and that this point lies in* f.

Proof. d and e must meet, since $\angle$s A and B are not straight $\angle$s. Call this point O. Draw $OK \perp AB$, $OL \perp BC$, and $OM \perp AC$. Then

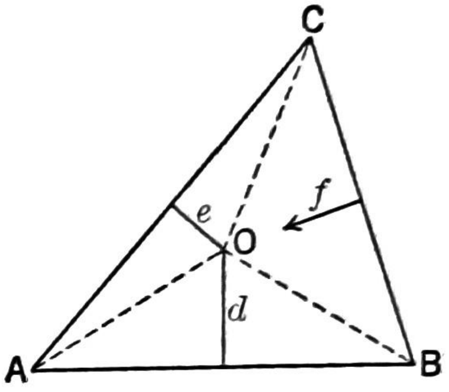
(1) $OK = OM$	(1) 79 (*a*).
(2) $OK = OL$	(2) Same reason.
(3) $\therefore OL = OM$	(3) Ax. 1.
(4) $\therefore O$ lies in f	(4) 79 (*b*).
(5) $\therefore d$, e, and f meet in point O, and this point is equidistant from AB, BC, and AC.	(5) We have shown that O lies in d, e, and f, and that $OK = OM = OL$

Q.E.D.

Remark: The point O is the center of the circle which will touch AB, BC, and AC, if OK is used as radius.

Proposition XXXVII. Theorem

122. The perpendicular bisectors of the sides of a triangle meet in a point which is equidistant from the vertices of the triangle.



Given $\triangle ABC$ with d, e, f the $\perp$ bisectors of AB, AC, and BC respectively.

Prove d, e, f meet in a point O which is equidistant from A, B, and C.

Plan is to show that d and e must meet in some point, and that this point lies in f.

Proof. d and e must meet, since $\angle A$ is not a straight $\angle$. Call this point O. Draw OA, OB, and OC. Then

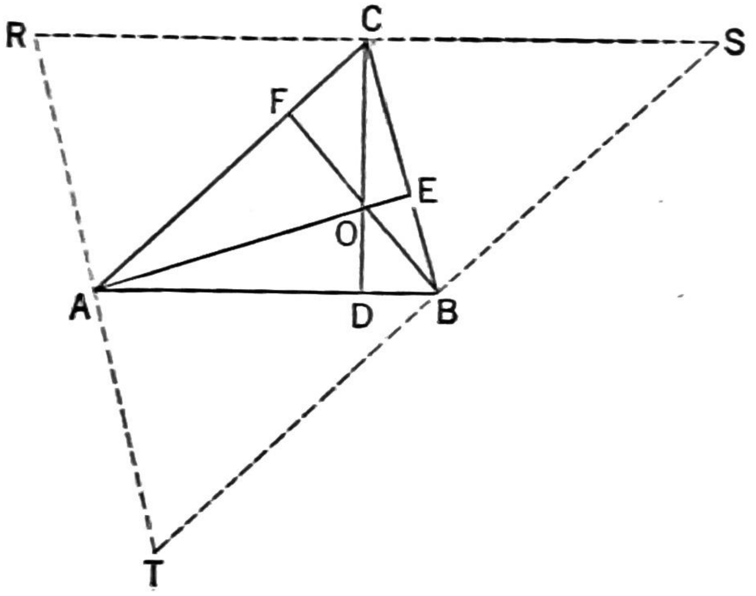
(1) $OA = OB$	(1) 77 *a*.
(2) $OA = OC$	(2) Same reason.
(3) $\therefore OB = OC$	(3) Ax. 1.
(4) $\therefore O$ lies in f	(4) If a point is equidistant from the ends of a line, it lies in the $\perp$ bisector of that line.
(5) $\therefore d$, e, and f meet in point O, and this point is equidistant from A, B, and C	(5) We have shown that O lies in d, e, and f, and that $OA = OB = OC$.

Q.E.D.

Remark: The point O is the center of the circle which will pass through A, B, and C, if OA is used as radius.

Proposition XXXVIII. Theorem

123. The altitudes of a triangle meet in a point.



Given $\triangle ABC$ with its altitudes CD, AE, and BF.

Prove CD, AE, and BF meet in a point O.

Analysis. If we can show that CD, AE, and BF are $\perp$ bisectors of the sides of another $\triangle$, our theorem is established. If such a $\triangle$ were drawn, its sides would have to be $\parallel$ to the sides of the given $\triangle$. (Why?)

Proof. Through A draw $RT \parallel BC$, through B draw $ST \parallel AC$, and through C draw $RS \parallel AB$. Then

(1) $RC = AB$	(1) 84.
(2) $CS = AB$	(2) Same reason.
(3) $\therefore RC = CS$	(3) Ax. 1.
(4) but $CD \perp RS$	(4) 60.
(5) $\therefore CD$ is the $\perp$ bisector of RS	(5) Def. of $\perp$ bisector.

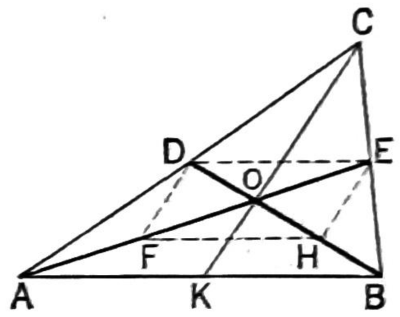
In like manner it can be shown that BF is the $\perp$ bisector of TS, and that AE is the $\perp$ bisector of RT.

(6) Hence CD, AE, and BF meet at O.	(6) 122.

Q.E.D.

Proposition XXXIX. Theorem

124. The medians of a triangle meet in a point which is two thirds of the distance from each vertex to the midpoint of the opposite side.



Given △ ABC with its medians AE, BD, and CK.

Prove AE, BD, and CK meet in a point O; and that $AO = \frac{2}{3} AE$, $BO = \frac{2}{3} BD$, and $CO = \frac{2}{3} CK$.

Analysis. We must first show that any *two medians such as* AE *and* BD *must meet at some point* O. *Then we can prove* $AO = \frac{2}{3} AE$ *if we can prove* $AO = 2\,OE$. *Also we must prove* $BO = 2\,OD$. *This suggests that we bisect* AO *and* BO *at* F *and* H *respectively, and try to prove* $FO = OE$ *and* $HO = OD$. *One way to prove this is to show that* FE *and* HD *can be made the diagonals of a* ▱, *viz.* $DEHF$.

Proof. Any two medians such as AE and BD must meet in some point O, for otherwise AE and BD would be ∥ and then ∡ EAB and DBA would be supplementary, which is absurd. Let F be the midpoint of AO and H the midpoint of BO. Draw FH, HE, ED, and DF. Then

(1) $FH \parallel AB$ and $FH = \frac{1}{2} AB$	(1) 111.
(2) $DE \parallel AB$ and $DE = \frac{1}{2} AB$	(2) Same reason.
(3) $\therefore FH \parallel DE$	(3) 61.
(4) and $FH = DE$	(4) Ax. 1.

(5) ∴ $FHED$ is a ▱	(5) 89.
(6) ∴ $FO = OE$ and $HO = OD$	(6) 90.
(7) but $AF = FO$ and $BH = HO$	(7) Const.
(8) ∴ $AF = FO = OE$ and $BH = HO = OD$ i.e. $AO = \frac{2}{3}\,AE$ and $BO = \frac{2}{3}\,BD$	(8) Ax. 1.
(9) Hence *any* median cuts off *any other* median two thirds of its length from the vertex.	(9) AE and BD were *any* two medians; also step 8.
(10) ∴ CK, the median from C, will cut off AO, two thirds of AE. In other words, CK will pass through O.	(10) Step 9. Q.E.D.

Remark: The point O is called the centroid of the triangle ABC. This point is the center of gravity of the triangle.

HOW TO ATTACK ORIGINALS

Helpful Suggestions for Solving Original Exercises

125. In the preceding pages the pupil has been given much training in analysis. This training should have developed considerable power to solve original exercises. After all, the real test of ability in geometry is the ease with which one can handle original questions. Before attempting the following exercises it is well to keep in mind the following:

(1) Read the question carefully and make a fairly large drawing. Do not use compasses unless required to do so.

(2) Make the drawing as general as possible. The figure should not represent a special case.

(3) Fix clearly in mind what is given and what is to be proved.

(4) If you do not know definitely how to proceed with the proof, use analysis.

(5) Be sure you make use of all the given facts.

(6) Two lines can be proved equal if you can prove them corresponding sides of congruent triangles, or equal legs of an isosceles triangle, or opposite sides of a parallelogram, or equal to lines that can be proved equal.

(7) Two angles can be proved equal if you can prove them corresponding angles of congruent triangles, or angles opposite equal sides in the same triangle, or opposite angles of a parallelogram, or alternate interior angles or corresponding angles of parallel lines, or supplements or complements of the same angle or of equal angles, or angles whose sides are parallel or perpendicular respectively.

(8) Two lines can be proved parallel if the alternate interior angles are equal, or if the corresponding angles are equal, or if the interior angles on the same side of the transversal are supplementary, or if they are both perpendicular to the same line, or if they are both parallel to the same line, or if they are the opposite sides of a parallelogram.

(9) Any two angles can be proved unequal if they are opposite unequal sides in the same triangle, or if one is an exterior angle and the other is an opposite interior angle of the same triangle, or if the conditions of Prop. XXXV are established, or by the axioms 8 through 13.

(10) Any two sides can be proved unequal if they are opposite unequal angles in the same triangle, or if the conditions of Prop. XXXIV are established, or by the axioms 8 through 13.

(11) Any quadrilateral can be proved a parallelogram if the opposite sides are parallel in pairs, or if the opposite sides are equal in pairs, or if two sides are equal and parallel, or if

the diagonals bisect each other, or if the opposite angles are equal in pairs.

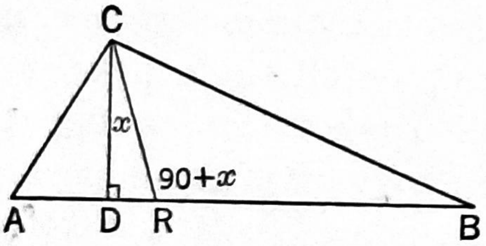
(12) If we are asked to prove one line the double of another, it is often helpful to prolong the shorter line its own length, or to bisect the longer line.

(13) If two lines or angles are given unequal, it is often helpful to mark off the less on the greater.

(14) Sometimes when other methods fail, try the indirect method of proof.

(15) At times the use of algebra is helpful.

Ex. 127.



Given $\triangle ABC$ with $CD \perp AB$, and CR the bisector of $\angle ACB$.

Prove $\angle DCR = \frac{1}{2}(\angle A - \angle B)$.

Proof. Denote $\angle DCR$ by $x°$. Then

(1) $\angle CRB = 90° + x°$	(1) 67.
(2) $90 + x + B + \frac{1}{2}C = 180°$	(2) 66.
(3) $\therefore B + \frac{1}{2}C = 90 - x$	(3) Ax. 3.
(4) but $A + \frac{1}{2}C = 90 + x$	(4) 67.
(5) $\therefore A - B = 2x$	(5) (4) − (3). Ax. 3.
(6) $\therefore x = \frac{1}{2}(A - B)$	(6) Ax. 5.

Q.E.D.

Miscellaneous Exercises

Ex. 128. The bisectors of two supplementary adjacent angles are perpendicular to each other.

Ex. 129. If the opposite angles of a quadrilateral are equal, the figure is a parallelogram.

Ex. 130. Prove the correctness of the following construction for bisecting ∠ ABC. On AB produced beyond B take $BD = BC$, and draw a line through $B \parallel DC$.

Ex. 131. Prove that the diagonals of a rhombus are unequal.

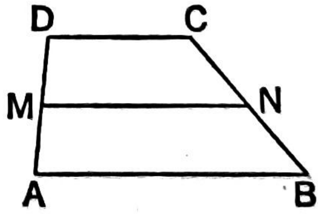
Ex. 132. If one of the legs of an isosceles triangle is produced through the vertex its own length, the line joining the end of the leg produced to the nearer end of the base is perpendicular to the base.

Ex. 133. The lines joining the midpoints of the sides of a triangle divide the triangle into four congruent triangles.

Ex. 134. The diagonals of a rhombus are perpendicular to each other, and bisect the angles of the rhombus.

Ex. 135. If the diagonals of a parallelogram are equal, the figure is a rectangle.

126. A *rectangle* is a parallelogram each of whose angles is a right angle.



127. A *trapezoid* is a quadrilateral which has only two sides parallel. Thus $ABCD$ is a trapezoid. The parallel sides AB and DC are called the *bases*. The line joining the midpoints of the nonparallel sides is called the *median* of the trapezoid. MN is the median. AD and BC are the *legs* of the trapezoid. If $AD = BC$, $ABCD$ is called an *isosceles trapezoid.*

Ex. 136. In an isosceles trapezoid each base makes equal angles with the legs.

Ex. 137. If the diagonals of a trapezoid are equal, the trapezoid is isosceles.

Ex. 138. In the quadrilateral $ABCD$, $AD = BC$ and $\angle D > \angle C$. Prove $\angle B > \angle A$.

Ex. 139. Perpendiculars drawn at the vertex of an isosceles triangle upon the legs inclose an angle equal to twice either base angle.

Ex. 140. Two angles whose sides are perpendicular, each to each, are either equal or supplementary.

Ex. 141. If one acute angle of a right triangle is 30°, the hypotenuse is twice the shorter leg.

Ex. 142. The midpoints of two opposite sides of a quadrilateral and the midpoints of the diagonals determine the vertices of a parallelogram.

Ex. 143. If two parallel lines are cut by a transversal, the bisectors of the interior angles form a rectangle.

Ex. 144. The bisectors of the angles of a rectangle inclose a square.

Ex. 145. If from any point in the base of an isosceles triangle parallels to the legs are drawn, a parallelogram is formed whose perimeter is constant. (This constant is equal to the sum of the legs of the triangle.)

Ex. 146. If from the ends of the base BC of an isosceles triangle ABC equal segments BD and CE are laid off on one leg and on the prolongation of the other leg, the line joining D and E is bisected by the base.

Sug. Draw $EH \parallel BA$, meeting BC produced at H.

Ex. 147. Two parallelograms are congruent if two sides and the included angle of one are equal respectively to two sides and the included angle of the other.

Sug. Superpose one figure upon the other.

Ex. 148. Two trapezoids are equal if their sides taken in order are equal each to each.

Sug. Use Ex. 147.

Ex. 149. ABC is a triangle. D is the foot of the perpendicular from A on BC, P is the midpoint of BC, X is a point on BC such that $XP = PD$. If the line through $P \perp BC$

meets AX in M, prove that $MB = MC$ and that $MX = MA$.

Ex. 150. Given an angle A and any point P within it. Show a method of drawing a line through P to the sides of the angle which shall be bisected at P.

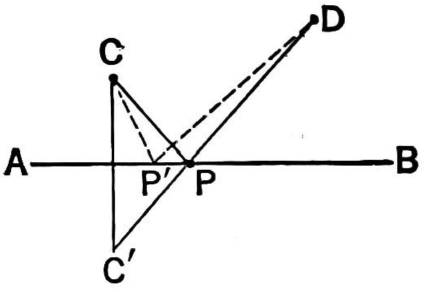
Ex. 151. Construct a line terminating in the sides of a given angle and equal and parallel to a given line.

Ex. 152. The bisectors of two exterior angles of a triangle include an angle equal to half the third exterior angle.

Ex. 153. The bisectors of the exterior angles at B and C of $\triangle ABC$ meet at D. Prove $\angle D = 90° - \frac{1}{2} \angle A$.

Ex. 154. Show how to bisect an angle formed by two converging lines, without producing the lines till they meet.

Ex. 155. Find a point in a line such that the sum of its distances from two fixed points on the same side of the line shall be the least possible.



Sug. Prove $CP + PD < CP' + P'D$.

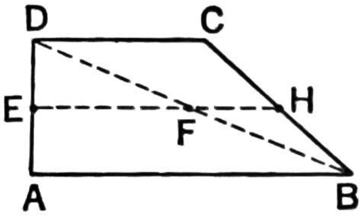
Ex. 156. Two triangles are congruent if the base, the median, and the altitude to the base of one triangle are equal respectively to the base, the median, and the altitude to the base of the other.

Ex. 157. The difference between the acute angles of a right triangle is equal to the angle between the median and the perpendicular drawn from the vertex of the right angle to the hypotenuse.

Ex. 158. ABC is an equilateral triangle. BD and CD are the bisectors of angles B and C. Prove that lines through D parallel to the sides AB and AC trisect BC.

Ex. 159. Three given straight lines meet in a point. Draw another straight line so that the two portions of it intercepted between the given lines shall be equal.

Sug. Prob. 150

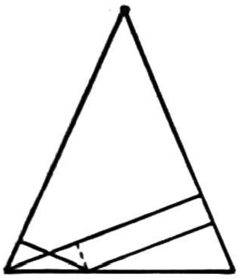


Ex. 160. The median of a trapezoid is parallel to the bases, and equals half the sum of the bases.

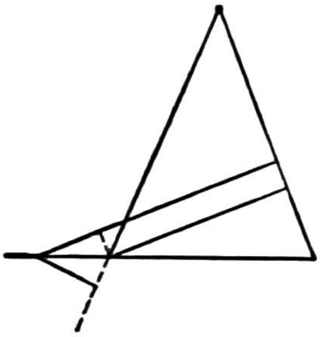
Sug. Draw DB. Let E be the midpoint of AD, F the midpoint of BD, H the midpoint of BC. Draw EF and FH, and prove that EFH is a continuous line. Apply 111.

Ex. 161. The median of a trapezoid passes through the midpoints of the two diagonals.

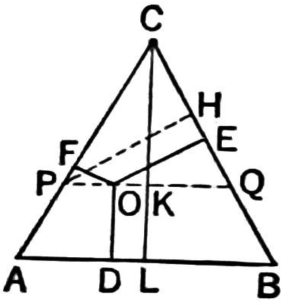
Ex. 162. The line joining the midpoints of the diagonals of a trapezoid is equal to half the difference of the bases.



Ex. 163. The sum of the perpendiculars from any point in the base of an isosceles triangle to the legs is equal to the altitude upon one of the legs.



Ex. 164. The difference of the distances from any point in the base produced of an isosceles triangle to the legs is equal to the altitude upon one of the legs.



Ex. 165. The sum of the perpendiculars drawn from any point within an equilateral triangle to the three sides is equal to the altitude of the triangle.

Sug. Through O draw $PQ \parallel AB$. Prove $OF + OE = PH$ (Ex. 163). Then prove $PH = CK$.

Ex. 166. $ABCD$ is a square. On the diagonal AC the point E is taken so that $AE = AB$. Through E a perpendicular to AE is drawn cutting BC in F. Prove $BF = EC$.

Ex. 167. The sum of the lines drawn from any point within a triangle to its vertices is less than the perimeter of the triangle but greater than the semiperimeter.

Sug. For the first part use Ex. 125.

Ex. 168. *AD*, *BE*, and *CF* are medians of △ *ABC*. *DH* is drawn equal and parallel to *BE*, and cuts *AC*. Prove that *HA* is equal and parallel to *CF*.

Ex. 169. The median to any side of a triangle is less than half the sum of the other two sides.

Sug. Prolong the median its own length.

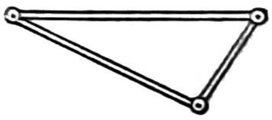
Ex. 170. A median of a triangle is greater than, equal to, or less than half of the side which it bisects, according as the angle opposite that side is acute, right, or obtuse.

Ex. 171. The bisectors of the angles of a quadrilateral form a second quadrilateral whose opposite angles are supplementary.

Ex. 172. The lines joining the midpoints of the sides of a rectangle, taken in order, inclose a rhombus.

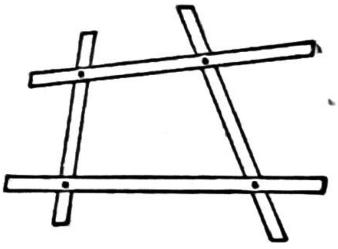
Practical Applications

Ex. 173. Three rods are hinged at their ends so as to form a triangle. Can the figure be made to change its shape? Will it be rigid?

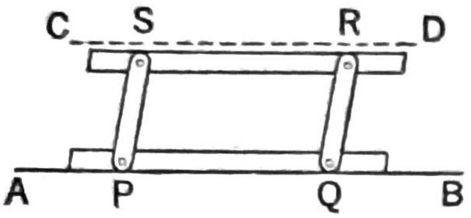


Ex. 174. Is a framework of four sticks rigid? How can a fifth be placed to make the framework rigid?

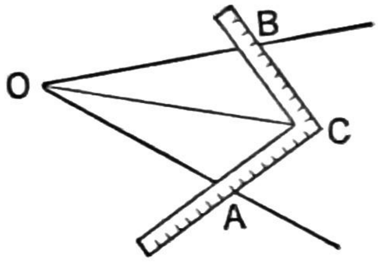
Note: In bridge building, and in structures generally, much use is made of the triangle as the unit of rigidity.



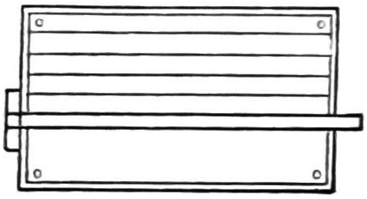
Ex. 175. The adjoining figure represents two rulers hinged at P, Q, R, S; $PQ = SR$, $PS = QR$. Explain how this framework can be used to draw lines parallel to a given line. Why will the lines be parallel?



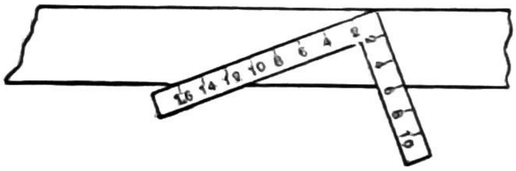
Ex. 176. A carpenter's steel square may be used to bisect an angle. Thus, mark off equal distances OA and OB on the sides of the angle. Place the square as shown. Mark point C, and join O and C. Prove that $\angle AOB$ is thus bisected.



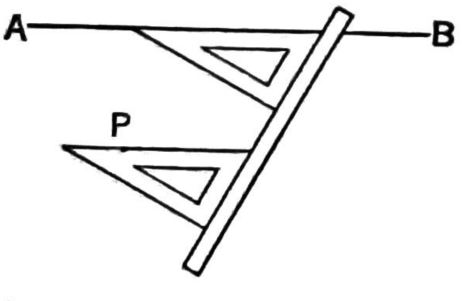
Ex. 177. Explain how the T-square may be used for drawing parallel lines. Why are the lines parallel?



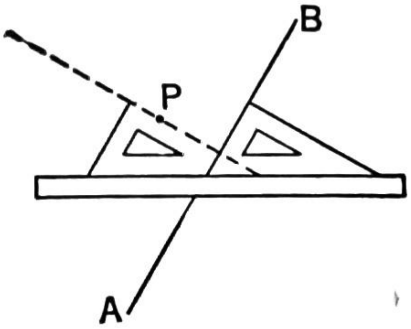
Ex. 178. A carpenter's steel square may be used to divide a board into strips of equal width. Thus to obtain four equal strips, place the square as shown, and make marks at the 4″, 8″, and 12″ divisions. Move the square and repeat this process. Draw lines through corresponding marks. Why will this divide the board into four equal strips?



Ex. 179. With triangle and ruler show how to draw a line through P parallel to AB.

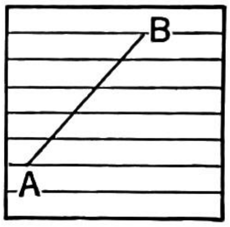


Solution. Place the triangle with its hypotenuse along AB. Then place the ruler along a side of the triangle, and slide the triangle along the ruler till its hypotenuse passes through P. Why will the line through P along the hypotenuse be parallel to AB?

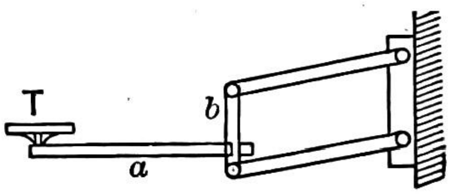


Ex. 180. With triangle and ruler show how to draw a line through P perpendicular to AB.

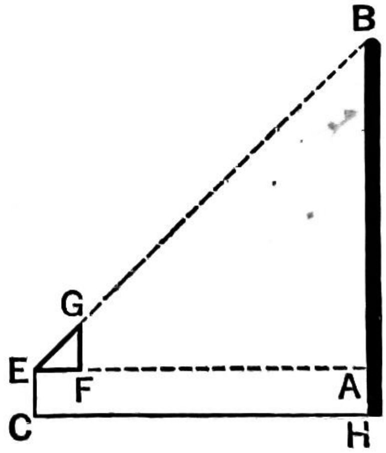
Solution. Place one side of the triangle along the line AB, place the ruler along the hypotenuse, and then slide the triangle along the ruler until the other side passes through P. Why will the line through P along this side be perpendicular to AB?



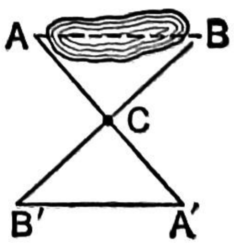
Ex. 181. Show how a given line may be divided into equal parts by use of ruled paper. Why is AB divided into five equal parts?



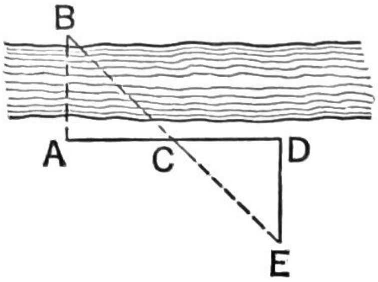
Ex. 182. Explain why the adjustable bracket here shown always carries the tray T in a horizontal position. (Note that bar a is $\perp$ bar b, T is fixed $\parallel$ a, and the wall is vertical.)



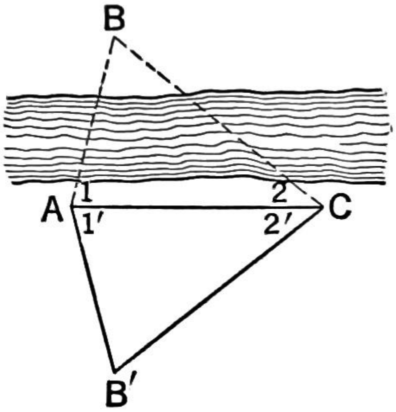
Ex. 183. Wishing to find the height of a pole a boy placed the right isosceles triangle EFG as here shown, and with his eye at E he sighted along EG to the top of the pole B. He then measured the distance CH. Prove that the height of the pole equals $CH + CE$.



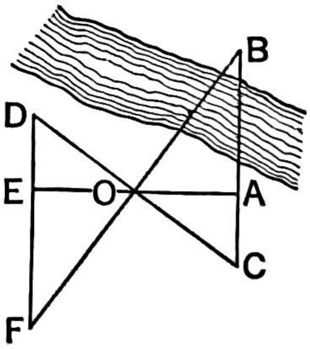
Ex. 184. To find the distance AB across a lake, proceed as follows: Set up a stake at any convenient point C from which A and B are visible. Set up a stake at A' in line with AC and so that $A'C = AC$. Do the same at B'. Measure $A'B'$. Prove that $A'B'$ is the required distance.



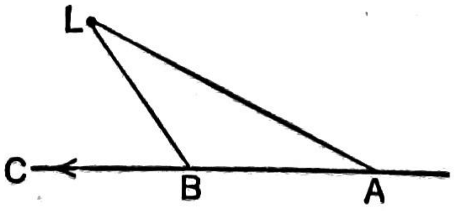
Ex. 185. To find the distance *AB* across a river, proceed as follows: Lay off the line *AD* at right angles to *AB*; set up a stake at *C*, midway between *A* and *D*; turn a right angle at *D*, and select a point *E* on this perpendicular so that *E*, *C*, and *B* are in line. Measure *DE*, and prove that *DE* is equal to the required distance.



Ex. 186. To find the distance from a point *A* to an inaccessible point *B* across a river, proceed as follows: Lay off any distance *AC*. Measure angles 1 and 2 (usually done by means of a transit). Make ∠ 1′ = ∠ 1 and ∠ 2′ = ∠ 2, and thus locate *B*′. Measure *AB*′, and prove that *AB*′ is equal to the required distance.



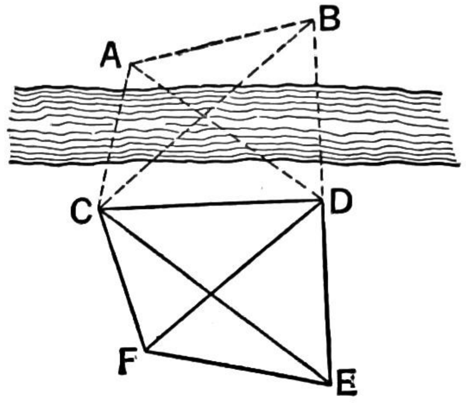
Ex. 187. To find the distance from a point *A* to an inaccessible point *B* across a river, proceed as follows: Set a stake at *C*, sighting it in line with *A* and *B*. Take *AC* any convenient distance. Select a point *O* from which *A*, *B*, and *C* are visible. Sight *D* in line with *O* and *C*, and make *DO* = *OC*. Sight *E* in line with *O* and *A*, and make *EO* = *OA*. Sight *F* in line with *E* and *D* and at the same time in line with *O* and *B*. Measure *EF*, and prove that *EF* is equal to the required distance.



Ex. 188. A ship is sailing in the direction *AC*, and a sailor wishes to calculate his distance from a lighthouse *L*. When the ship is at *A* he measures ∠ *BAL*,

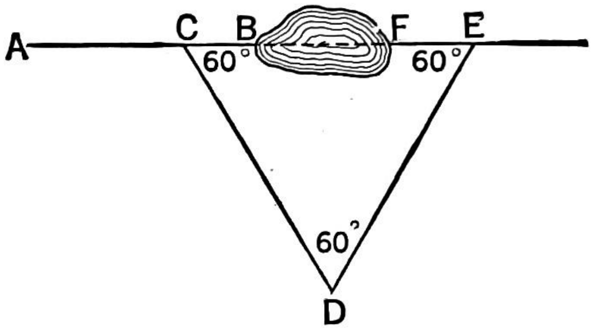
and when the ship is at B he finds that $\angle CBL$ is twice $\angle A$. From his log he finds the distance AB. Prove $BL = AB$. (This method is known as "doubling the angle on the bow.")

Ex. 189. To find the distance between two points A and B, when both are inaccessible, proceed as follows: Lay off $\triangle CFD \cong \triangle CAD$, also $\triangle CED \cong \triangle CBD$. Measure FE, and prove that FE equals AB.

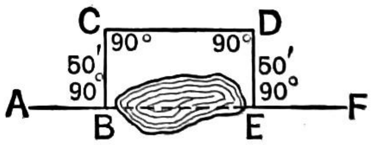


Sug. Prove that quadrilateral $ACDB$ can be made to coincide with quadrilateral $CDEF$.

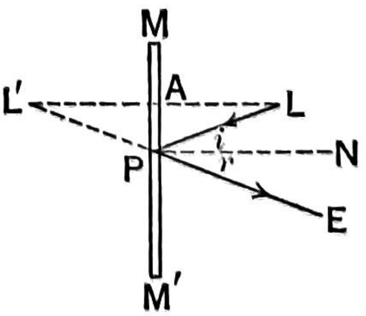
Ex. 190. To produce a line AB beyond an obstacle, proceed as follows: Select a convenient point C on AB, turn an angle of 60° and measure CD. At D turn an angle of 60° and measure DE equal to CD. At E turn an angle of 60°. Prove that FE is the prolongation of AB.

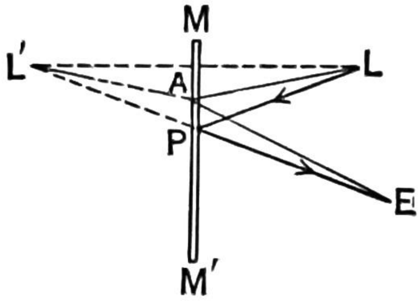


Ex. 191. In the adjoining figure, the line AB is prolonged by the method of right-angle offsets. Prove that EF is the prolongation of AB.



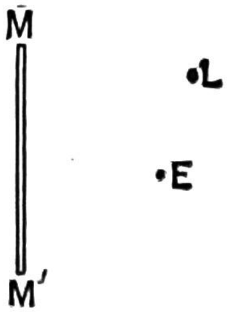
Ex. 192. If a ray of light LP strikes a plane mirror MM' at P it is reflected along PE so that $\angle i = \angle r$ ($PN \perp MM'$). EP prolonged meets LA prolonged at L' ($LA \perp MM'$). Prove $L'A = LA$, i.e. the light appears to be the same distance behind the mirror as it is actually in front of it.



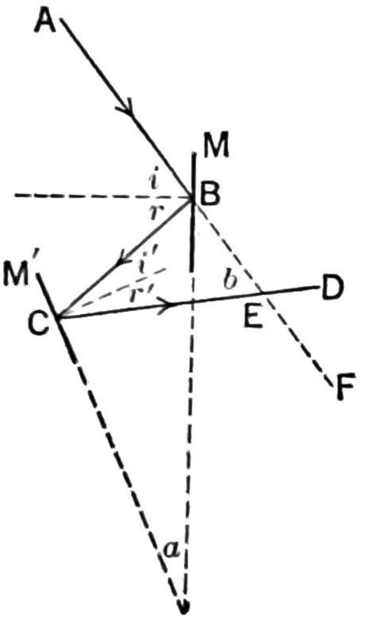


Ex. 193. Prove that light is reflected along the shortest path.

Sug. Let A be *any* point on MM' other than P. Prove $LP + PE < LA + AE$.



Ex. 194. A ray of light from L strikes a plane mirror MM' and reaches the eye at E. Locate the point where it strikes the mirror.



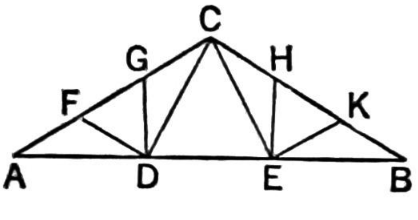
Ex. 195. The sextant is an instrument that measures the angle between two distant objects. It is constructed on the principle that if a ray of light is reflected successively between two plane mirrors, the angle between the first and last direction of the ray is twice the angle between the mirrors. In the adjoining figure prove $\angle FED = 2 \angle a$.

Outline of proof.

$\angle b = (i + r) - (i' + r') = 2r - 2r'$

$\angle a = (90° - i') - (90 - r) = r - r'$

$\therefore \angle b = \angle FED = 2 \angle a$



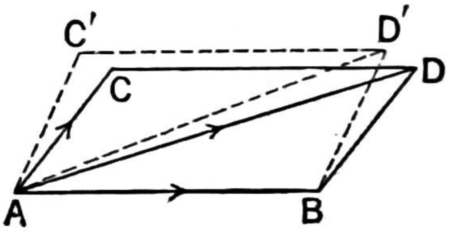
Ex. 196. Construct the fan truss shown in the adjoining figure. ABC is an isosceles triangle. $AD = DC = CE = EB$; $AF = FD = DG = GC$; $BK = KE = EH = HC$. Prove your construction.

Ex. 197. If a point possesses two simultaneous velocities represented by two straight lines AB, AC, their resultant is represented in magnitude and direction by the diagonal *AD* of the parallelogram constructed on *AB* and *AC* as adjacent sides. This is known as the parallelogram of velocities. Show that the resultant of two velocities *AB*, *AC* is represented by 2 *AM* when *M* is the midpoint of *BC*.

Ex. 198. *ABCD* is a square, and *O* is the midpoint of *BC*. Show that the resultant of the velocities *AB*, *AO*, *AC* is 3 *AO*.

Ex. 199. A particle has a velocity of 5 ft. per second, and an equal velocity at an angle of 120° is communicated to it. Show that the final velocity is 5 ft. per second, and that we have an example of change of direction without change of speed.

Ex. 200. If two forces F_1 and F_2 acting on a particle (or at a point *A*) be represented by two lines *AB*, *AC*, their resultant *R* is represented by the diagonal *AD* of the parallelogram *ABDC* constructed on *AB* and *AC* as adjacent sides. This principle is called the parallelogram of forces. Prove that the value of the resultant decreases as the angle between the forces increases.



Ex. 201. Two equal forces *F* and *F′* act at an angle of 120°. Find their resultant.

Ex. 202. If a force is resolved into two components, prove that the greater component always makes the smaller angle with the force.

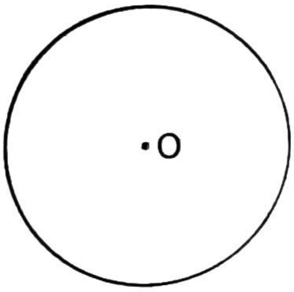
Ex. 203. The pull on the rope of a canal-boat is 100 pounds, and the direction of the rope makes an angle of 60° with the parallel banks. Find the force urging the boat forward.

BOOK II

THE CIRCLE

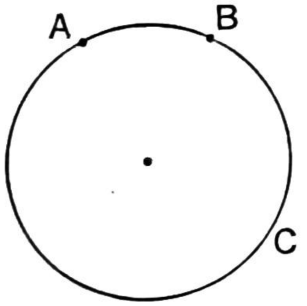
Fundamental Definitions

128. A *circle* is a plane closed curve such that all its points are equally distant from a fixed point in the plane. The fixed point is called the *center* of the circle. Compasses are used to draw circles. Note that a circle is a curved *line.*



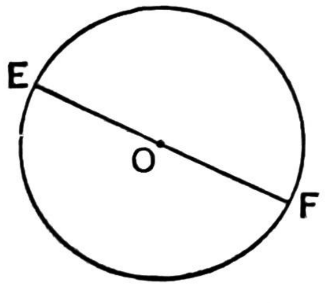
129. The length of the entire circle is called the *circumference.*

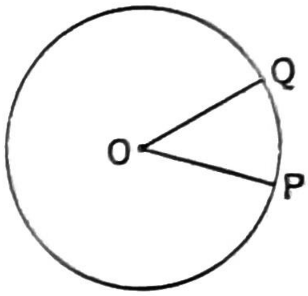
130. Any portion of a circle is called an *arc.* A *semicircle* is an arc that is half of a circle. A *minor arc* is an arc that is less than a semicircle. A *major arc* is one that is greater than a semicircle. Thus in the annexed drawing, arc AB is a minor arc, arc ACB is a major arc.



131. A *radius* of a circle is a straight line from the center of the circle to any point of the circle. (The plural of radius is *radii.*)

A *diameter* of a circle is a straight line passing through the center of the circle and terminating in the circle. Thus OF is a radius, EF is a diameter. A diameter of any circle is therefore twice a radius of that circle.



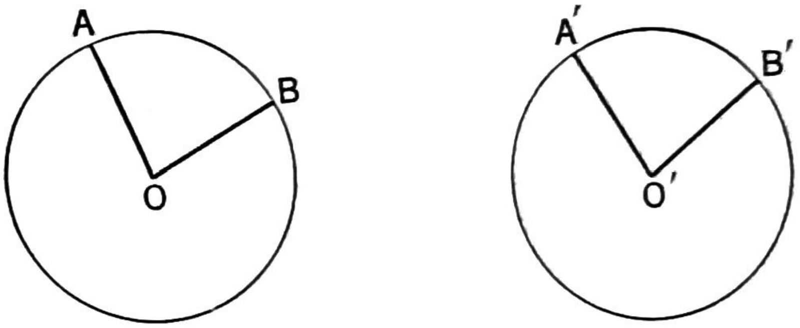


132. A *central angle* is an angle formed by two radii; e.g. ∠ POQ. We say central angle POQ has the arc PQ; and conversely, arc PQ has central angle POQ. We might also say that ∠ POQ intercepts $\overset{\frown}{PQ}$, and that $\overset{\frown}{PQ}$ subtends ∠ POQ.

CENTRAL ANGLES

Proposition I. Theorem

133. In the same circle or in equal circles, equal central angles have equal arcs.



Given the equal ⓢ O and O' with central ∠ AOB = central ∠ $A'O'B'$.

Prove $\overset{\frown}{AB} = \overset{\frown}{A'B'}$.

Plan is to place one figure upon the other, and thus show that the arcs AB and A′B′ can be made to coincide.

Proof. Place ⊙ O upon ⊙ O' so that ∠ AOB coincides with its equal, ∠ $A'O'B'$. Then as a result of this placing

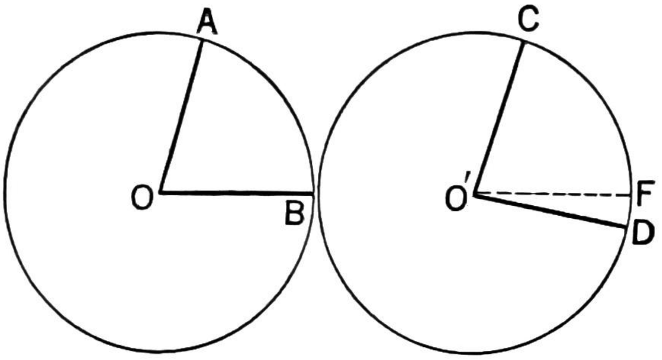
(1) A will fall on A' and B will fall on B'	(1) $OA = O'A'$ and $OB = O'B'$ because radii of equal ⓢ are equal.
(2) ∴ $\overset{\frown}{AB}$ coincides with $\overset{\frown}{A'B'}$	(2) All points of each arc are equidistant from O', be-

	cause the ⓢ are given equal, and O is on O'.
(3) $\therefore \overset{\frown}{AB} = \overset{\frown}{A'B'}$	(3) If two lines can be made to coincide throughout, the lines are said to be equal.

Q.E.D.

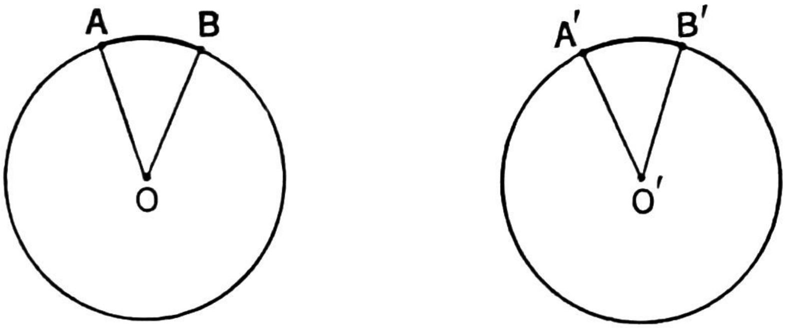
134. *Cor.* In the same ⊙ or in equal ⓢ, the greater of two central angles has the greater arc.

Outline of proof. Construct $\angle CO'F$ equal to $\angle AOB$. Then $\overset{\frown}{CD} > \overset{\frown}{CF}$, but $\overset{\frown}{CF} = \overset{\frown}{AB}$. $\therefore \overset{\frown}{CD} > \overset{\frown}{AB}$.



Proposition II. Theorem

135. Conversely, in the same circle or in equal circles, equal arcs have equal central angles.



Given the equal ⓢ O and O' with $\overset{\frown}{AB} = \overset{\frown}{A'B'}$.

Prove $\angle AOB = \angle A'O'B'$.

Plan is to place one figure upon the other, and thus show that the angles AOB and A'O'B' can be made to coincide.

Proof. Place ⊙ O upon ⊙ O' so that OA coincides with its equal $O'A'$ and $\overset{\frown}{AB}$ with its equal $\overset{\frown}{A'B'}$. Then as a result of this placing

(1) OB will coincide with $O'B'$	(1) Only one st. line can be drawn connecting two points.
(2) $\therefore \angle AOB$ will coincide with $\angle A'O'B'$	(2) Their respective vertices and sides coincide.
(3) $\therefore \angle AOB = \angle A'O'B'$	(3) Definition of equal angles. Q.E.D.

136. *Cor.* In the same ⊙ or in equal ⓢ, the greater of two arcs has the greater central angle.

Outline of proof.

Lay off $\overset{\frown}{AC} = \overset{\frown}{A'B'}$. Draw OC

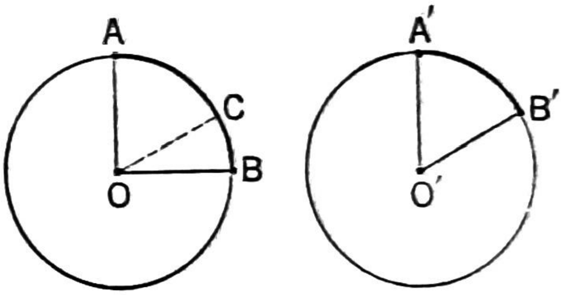
Then $\overset{\frown}{AB} > \overset{\frown}{AC}$

$\therefore OC$ falls within $\angle AOB$

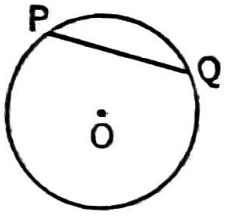
$\therefore \angle AOB > \angle AOC$

but $\angle AOC = \angle A'O'B'$

$\therefore \angle AOB > \angle A'O'B'$.



137. A *chord* of a ⊙ is a straight line whose end points are on the ⊙. Thus PQ is a chord of ⊙ O.



State whether the following statements are true or false, and give reasons.

(*a*) If AB and CD are two diameters of ⊙ O, then $\overset{\frown}{AD} = \overset{\frown}{BC}$, and $\overset{\frown}{AC} = \overset{\frown}{BD}$.

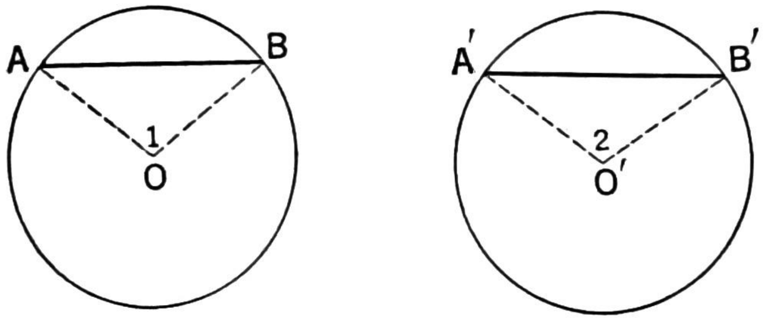
(*b*) AB is a diameter and AC is a chord of ⊙ O. Then $\angle BOC$ is twice $< BAC$.

(*c*) $\overset{\frown}{PQ}$ of ⊙ O is greater than $\overset{\frown}{P'Q'}$ of ⊙ O'. Hence $\angle POQ$ is greater than $\angle P'O'Q'$.

CHORDS AND ARCS

Proposition IIIa. Theorem

138. In the same circle or in equal circles, equal chords have equal arcs.



Given the equal Ⓢ O and O' with chord AB = chord $A'B'$.

Prove $\overset{\frown}{AB} = \overset{\frown}{A'B'}$.

Analysis. We can prove the arcs equal if we can prove their central angles equal; and we can prove these angles equal if we can prove them corresponding parts of congruent △.

Proof. Draw OA, OB, $O'A'$, and $O'B'$. Then

(1) $OA = OB = O'A' = O'B'$	(1) Radii of equal Ⓢ are equal.
(2) $AB = A'B'$	(2) Given.
(3) $\therefore \triangle OAB \cong \triangle O'A'B'$	(3) s. s. s. = s. s. s.
(4) $\therefore \angle 1 = \angle 2$	(4) Corr. parts of congruent △ are equal.
(5) $\therefore \overset{\frown}{AB} = \overset{\frown}{A'B'}$	(5) 133. Q.E.D.

How can you prove major arc AB = major arc $A'B'$?

Proposition IIIb. Theorem

139. Conversely, in the same circle or in equal circles, equal arcs have equal chords.

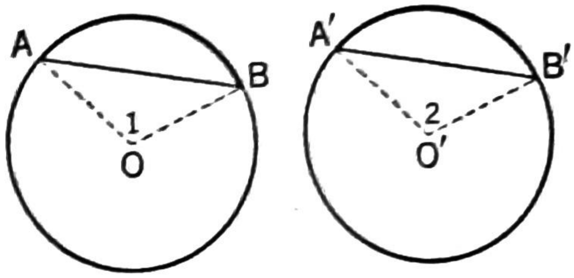
Outline of proof.

$\widehat{AB} = \widehat{A'B'}$

$\therefore \angle 1 = \angle 2$

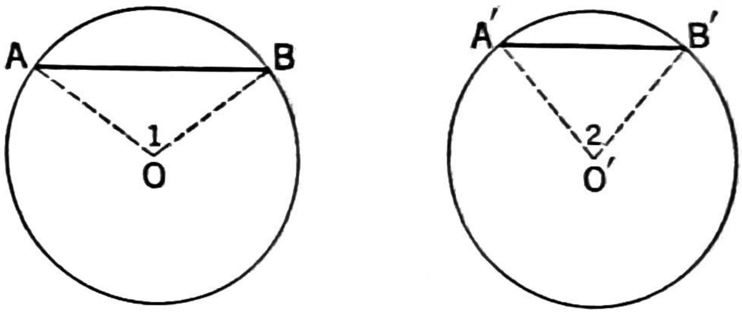
$\therefore \triangle AOB \cong \triangle A'O'B'$

$\therefore \quad AB = A'B'.$



Proposition IVa. Theorem

140. In the same circle or in equal circles, the greater of two chords has the greater minor arc.



Given the equal ⓢ O and O' with chord $AB >$ chord $A'B'$.

Prove $\widehat{AB} > \widehat{A'B'}$.

Analysis. We can prove $\widehat{AB} > \widehat{A'B'}$*, if we can prove the former has the greater central angle. That we can do, because in the* $\triangle$ AOB *and* $A'O'B'$, $AB > A'B'$.

Proof. Draw OA, OB, $O'A'$, and $O'B'$. Then in $\triangle$ AOB and $A'O'B'$

(1) $OA = OB = O'A' = O'B'$	(1) Radii of equal ⓢ are equal.
(2) $AB > A'B'$	(2) Given.
(3) $\therefore \angle 1 > \angle 2$	(3) 120.
(4) $\therefore \widehat{AB} > \widehat{A'B'}$	(4) 134.

Q.E.D.

Proposition IVb. Theorem

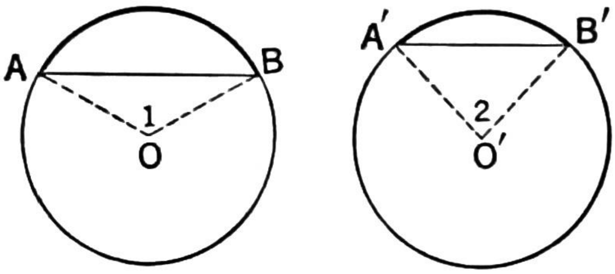
141. Conversely, in the same circle or in equal circles, the greater of two minor arcs has the greater chord.

Outline of proof.

$\overset{\frown}{AB} > \overset{\frown}{A'B'}$

$\therefore \angle 1 > \angle 2$

$\therefore AB > A'B'$.



GROUP VI

State whether the following statements are true or false, and give reasons.

(*a*) Equal chords have equal arcs.

(*b*) Two circles are equal if they have equal chords.

(*c*) The radius that bisects an arc is the perpendicular bisector of the chord of the arc.

(*d*) Two circles having the same point as center are equal.

(*e*) Only one circle can be drawn through two given points.

(*f*) A, a point on $\odot O$, is equidistant from the sides of central $\angle BOC$. Hence A is the midpoint of $\overset{\frown}{BC}$.

(*g*) If AB and BC are equal chords of a circle, the bisector of $\angle ABC$ must pass through the center of the circle.

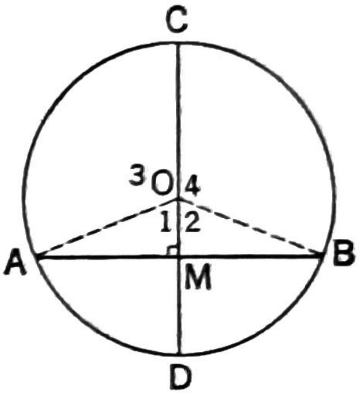
(*h*) If AB and CD are two diameters of $\odot O$, then chord $AC \parallel$ chord BD.

(*i*) $\angle ABC$ is inscribed in a circle, and AB is a diameter. If DE is a diameter $\parallel CB$ then D is the midpoint of $\overset{\frown}{AC}$.

(*j*) A line bisecting a chord of a circle will also bisect its arc.

Proposition V. Theorem

142. If a diameter is perpendicular to a chord, it bisects the chord and its two arcs.



Given $\odot O$ with CD a diameter $\perp$ chord AB at M.

Prove $AM = MB$, $\widehat{AD} = \widehat{BD}$, $\widehat{AC} = \widehat{BC}$.

Analysis. We can prove $AM = MB$ *by congruent* $\triangle$*, and we can prove the arcs equal by showing that their central angles are equal.*

Proof. Draw OA and OB. Then

(1) $OA = OB$	(1) Radii of the same $\odot$ are equal.
(2) $OM = OM$	(2) Iden.
(3) $\therefore \triangle OMA \cong \triangle OMB$	(3) 75.
(4) $\therefore AM = MB$ and $\angle 1 = \angle 2$	(4) Corr. parts of congruent $\triangle$ are equal.
(5) $\therefore \widehat{AD} = \widehat{BD}$	(5) 133.
(6) $\angle 3 = \angle 4$	(6) Supplements of equal angles are equal.
(7) $\therefore \widehat{AC} = \widehat{BC}$	(7) 133. Q.E.D.

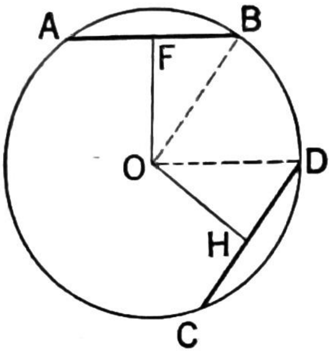
143. *Cor.* The perpendicular bisector of a chord passes through the center of the circle.

Outline of proof. Since O is equidistant from A and B, it must lie in the $\perp$ bisector of AB; i.e. the $\perp$ bisector of AB passes through O.

EQUAL CHORDS

Proposition VIa. Theorem

144. In the same circle or in equal circles, equal chords are equidistant from the center of the circle.



Given $\odot O$ with $AB = CD$.

Prove O is equidistant from AB and CD.

Analysis. The distance of a point from a line is measured on a perpendicular from the point to the line. We must therefore draw $OF \perp AB$ *and* $OH \perp CD$, *and then prove* $OF = OH$ *by proving them corresponding parts of congruent* $\triangle$s.

Proof. Draw $OF \perp AB$, and $OH \perp CD$. Draw OB and OD. Then

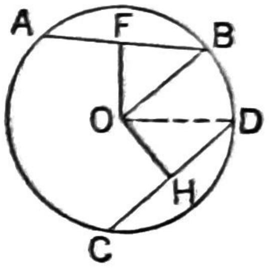
(1) $AB = CD$	(1) Given.
(2) $\therefore \frac{1}{2} AB = \frac{1}{2} CD$	(2) Halves of equals are equal.
(3) but $FB = \frac{1}{2} AB$ and $HD = \frac{1}{2} CD$	(3) 142.
(4) $\therefore FB = HD$	(4) Sub.
(5) $OB = OD$	(5) Radii of the same $\odot$ are equal.
(6) $\therefore \triangle OFB \cong \triangle OHD$	(6) 75.
(7) $\therefore OF = OH$	(7) Corr. parts of congruent $\triangle$s are equal.

Q.E.D.

Proposition VIb. Theorem

145. Conversely, in the same circle or in equal circles, chords which are equidistant from the center of the circle are equal.

Outline of proof.
$OF = OH$
$OB = OD$
$\therefore \triangle OFB \cong \triangle OHD$
$\therefore FB = HD$
$\therefore 2\,FB = 2\,HD$
$\therefore AB = CD$



146. In the same ⊙ or in equal Ⓢ two arcs can be proved equal if (*a*) we can prove their central angles equal; or (*b*) we can prove their chords equal.

147. In the same ⊙ or in equal Ⓢ two chords can be proved equal if (*a*) we can prove their arcs equal; or (*b*) we can prove the chords are equidistant from the center of the ⊙.

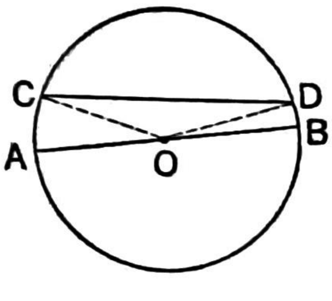
148. In the same ⊙ or in equal Ⓢ two arcs can be proved unequal by 134 or by 140.

149. In the same ⊙ or in equal Ⓢ two chords can be proved unequal by 141 or by 153.

Exercises

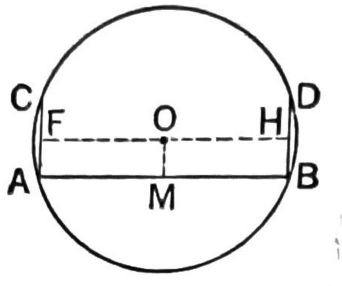
Ex. 204. The diameter of a circle is greater than any other chord.

Sug. Draw OC and OD. Then $OC + OD > CD$.



Ex. 205. The line drawn from the midpoint of a chord to the midpoint of its arc is perpendicular to the chord.

Ex. 206. Two chords drawn perpendicular to a third chord at its extremities are equal.



Sug. Through O draw $FH \perp AC$, and $OM \perp AB$. Then prove $OF = OH$, etc.

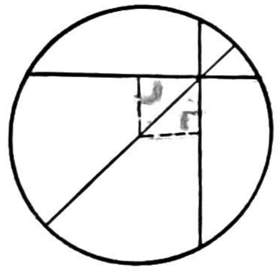
Ex. 207. Prove that if two diameters of a circle are perpendicular to each other and have their ends connected, a square is formed.

Ex. 208. Two radii form an acute angle at the center of a circle. Extend one of the radii through the center of the circle, and prove that the chords of the arcs intercepted by the central angles are unequal.

Ex. 209. If two concentric circles are cut by a line, the segments of the line between the circles are equal.

150. *Concentric* circles are circles that have the same center.

Ex. 210. If two intersecting chords make equal angles with the diameter through the point of intersection, the two chords are equal.



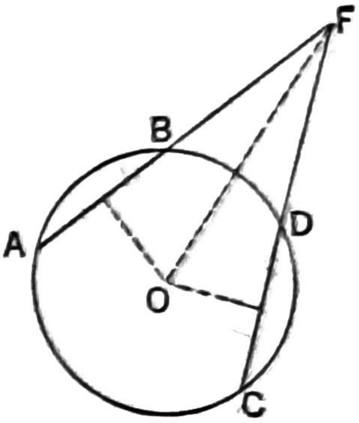
Ex. 211. A straight line is drawn cutting two equal intersecting circles and parallel to the line connecting their centers. Prove that the chords intercepted in the circles are equal.

Ex. 212. The diagonals of an equilateral pentagon inscribed in a circle are equal.

151. If the vertices of any polygon are on a circle, the polygon is said to be *inscribed* in the circle. The circle is then said to be *circumscribed* about the polygon.

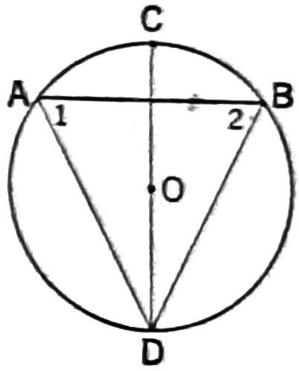
Ex. 213. Bisect a given arc.

Ex. 214. In the annexed drawing, the equal chords AB and CD are produced to meet at F. Prove $FA = FC$.

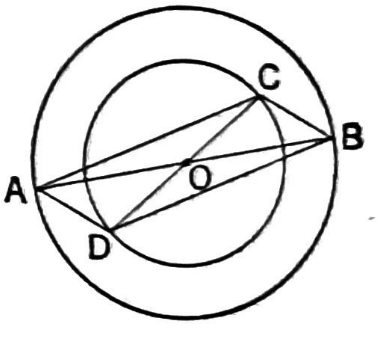


Ex. 215. A line joining the midpoints of two parallel chords passes through the center of the circle.

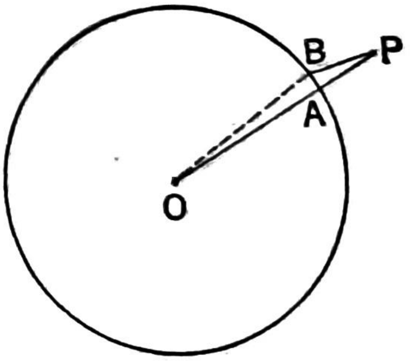
Ex. 216. In the annexed drawing, AB is any chord in $\odot O$. CD is a diameter drawn to the midpoint of arc AB. Prove $\angle 1 = \angle 2$.



Ex. 217. AB and CD are intersecting diameters of two concentric circles. Prove that A, C, B, and D are the vertices of a parallelogram.

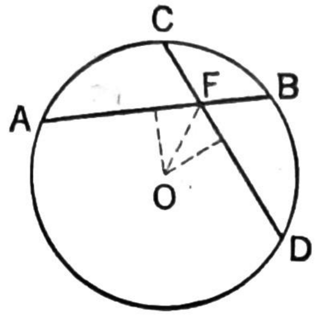


Ex. 218. Prove that the shortest distance from a point to a circle is measured along the line connecting that point with the center of the circle.



Sug. Prove $PA < PB$, any other line from P to the $\odot$. $PA + AO < PB + BO$. $\therefore PA < PB$.

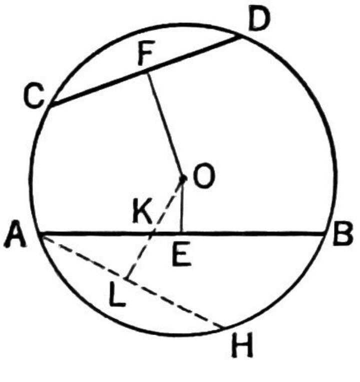
Ex. 219. Prove that if two equal chords intersect within a circle, their corresponding segments are equal.



UNEQUAL CHORDS

Proposition VII. Theorem

152. In the same circle or in equal circles, the greater of two chords is at the less distance from the center of the circle.



Given $\odot O$ with $AB > CD$.

Prove the distance from O to $AB <$ the distance from O to CD.

Analysis. We must prove the $\perp$ *from* O *to* $AB <$ the $\perp$ from O to CD, *i.e. we must prove* $OE < OF$ *(where* $OE \perp AB$, *and* $OF \perp CD$*). This we can do, if we can prove* $OE <$ *some line which equals* OF*. We can get a line equal to* OF*, if we can get a chord equal to* CD*.*

Proof. Draw $OE \perp AB$, and $OF \perp CD$. Since $AB > CD$, $\overset{\frown}{AB} > \overset{\frown}{CD}$. (Why?) $\therefore$ if beginning at A we lay off $\overset{\frown}{AH} = \overset{\frown}{CD}$, point H must fall somewhere between A and B. Draw AH, also draw $OL \perp AH$. Then

(1) $\overset{\frown}{AH} = \overset{\frown}{CD}$	(1) Const.
(2) $\therefore AH = CD$	(2) 139.
(3) $\therefore OL = OF$	(3) 144.
(4) but $OL > OK$	(4) Why?
(5) and $OK > OE$	(5) Why?
(6) $\therefore OL > OE$	(6) Why?
(7) $\therefore OF > OE$	(7) Why?
or in other words, $OE < OF$	
	Q.E.D.

GROUP VII

Complete each of the following statements:

(*a*) AB and CD are two equal chords of the same circle. If AB is 8″ from the center, then CD is . . . inches from the center.

(*b*) The radius of $\odot\ O$ is 6″, and central $\angle\ AOB$ is 60°. Chord AB is . . . inches.

(*c*) AB is a chord of $\odot\ O$, and the diameter $CD \perp AB$. If $\overset{\frown}{AC}$ is 40°, then $\overset{\frown}{BC}$ is . . . degrees.

(*d*) In $\odot\ O$ chord $AB = 5''$ and chord $CD = 3''$. Hence chord CD is at the . . . distance from the center of the circle.

(*e*) If the extremities of two diameters of a $\odot$ are joined, the figure formed is a . . . .

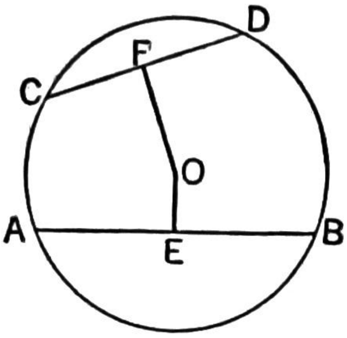
(*f*) If two circles whose centers are O and O' intersect in A and B, then $\angle\ OAO'$ . . . $\angle\ OBO'$.

(*g*) In the above example, if the common chord AB is drawn, then OO' is the . . . of AB.

(*h*) In $\odot O$, $\overset{\frown}{AB} > \overset{\frown}{CD}$. Hence chord AB is . . . the center of the $\odot$ than chord CD.

Proposition VIII. Theorem (*Converse of VII*)

153. In the same circle or in equal circles, if two chords are unequally distant from the center, the chord at the less distance is the greater.



Given ⊙ O with $OE < OF$ (where $OE \perp AB$, and $OF \perp CD$).

Prove $AB > CD$.

Analysis. If we can show that it is impossible for AB to equal CD or for AB to be less than CD, then AB must be greater than CD. Hence

Plan is to use the indirect method of proof.

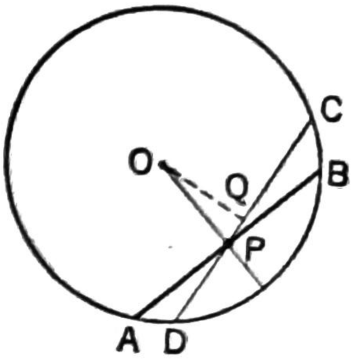
Proof. AB is either equal to CD, less than CD, or greater than CD.

(1) If $AB = CD$
then $OE = OF$ (144)
but $OE < OF$ (Given)
∴ the first supposition leads to a contradiction of what is given.

(2) If $AB < CD$
then $OE > OF$ (152)
but $OE < OF$ (Given)
∴ the second supposition leads to a contradiction of what is given.

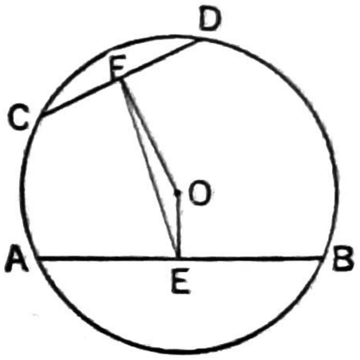
Hence there is only one possibility left, that is
$AB > CD$. Q.E.D.

Exercises

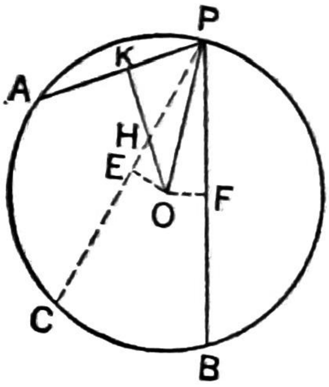


Ex. 220. The shortest chord that can be drawn through a point within a circle is the chord that is perpendicular to the radius drawn through the point.

Sug. Let CD be any other chord through P. Since $OQ < OP$, $CD > AB$.



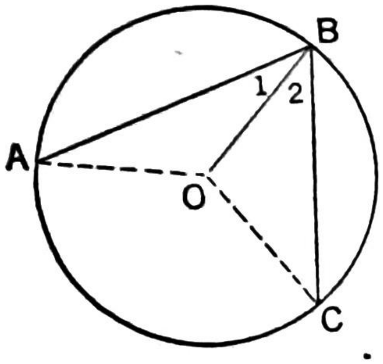
Ex. 221. In the annexed drawing, if $OE \perp AB$, and $OF \perp CD$, and $\angle OEF > \angle OFE$ prove that $AB > CD$.



Ex. 222. If from a point on a circle two chords are drawn making unequal angles with the radius drawn to that point, then the chords are unequal.

Sug. Make $\angle OPC = \angle OPB$. Then $OE = OF$. Since $OK > OH > OE$, then $OK > OF$, etc.

Ex. 223. If an equilateral triangle and a square are inscribed in a circle, the sides of the triangle are nearer to the center than the sides of the square.



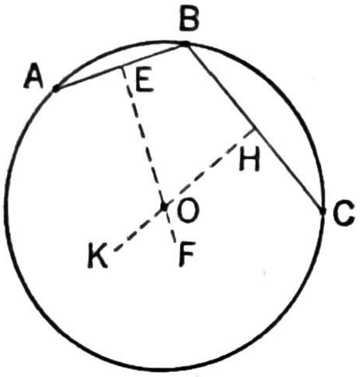
Ex. 224. If two unequal chords intersect on a circle, the greater chord makes the less angle with the radius through the point of intersection.

Sug. $\angle AOB > \angle BOC$; $\therefore 180° - \angle AOB < 180 - \angle BOC$; $\therefore \angle 1 < \angle 2$.

CIRCLE DETERMINED

Proposition IX. Theorem

154. Through three points not in a straight line, one circle and only one can be drawn.



Given three points A, B, and C not in a straight line.

Prove one circle, and only one, can be drawn through A, B, and C.

Analysis. We must locate a point that is equidistant from A, B, and C. Such a point must be in the ⊥ *bisector of AB, and also in the* ⊥ *bisector of BC. Hence it must be the intersection of these two* ⊥ *bisectors.*

Proof. At E, the midpoint of AB, draw $EF \perp AB$. At H, the midpoint of BC, draw $HK \perp BC$. Then

(1) EF and HK will intersect in some point O	(1) AB and BC are given not in the same straight line.
(2) $OA = OB$ and $OB = OC$	(2) 77 *a*.
(3) $\therefore OA = OB = OC$	(3) Ax. 1.
(4) ∴ if a ⊙ were drawn with O as center and OA as radius it would	(4) $OA = OB = OC$.

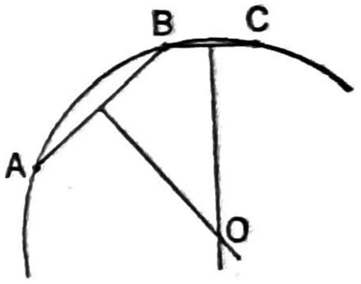
pass through A, B, and C	
(5) And it is the only ⊙ that can pass through A, B, and C	(5) EF is the only ⊥ bisector of AB. HK is the only ⊥ bisector of BC, and EF and HK can intersect at only one point.

Q.E.D.

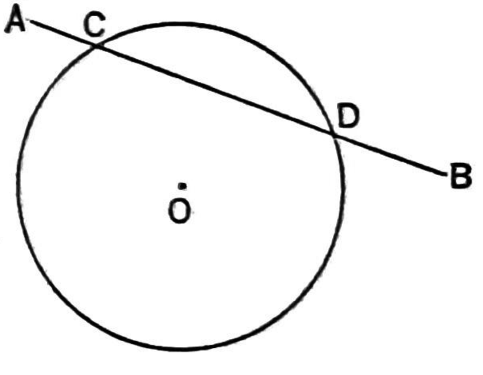
Note. Proposition IX may be stated as follows: Three points not in a straight line determine a circle.

155. Since A, B, and C determine a triangle, Prop. IX gives us a method of circumscribing a circle about a triangle. The point O is called the *circum-center* of △ ABC.

156. Prop. IX also enables us to find the center of a given arc of a circle. Thus take any three points on the arc, such as A, B, C. Draw the ⊥ bisector of AB and of BC. Where these meet, O, is the center of the arc.

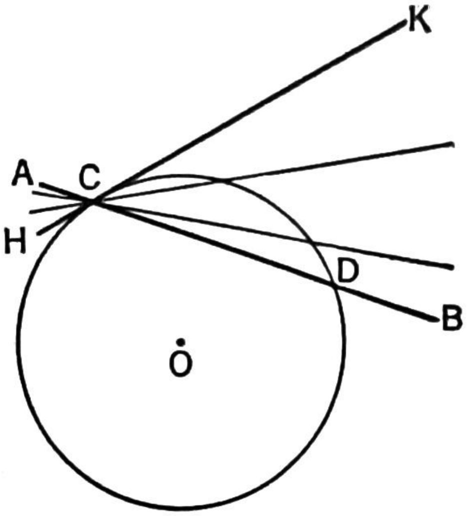


157. A *secant* is an unlimited straight line that cuts a circle in two points. Thus AB is a secant of ⊙ O. It cuts the circle in points C and D.



158. Imagine that the secant AB turns about C, as indicated in the drawing (top of next page). Points C and D get closer and closer, until finally D coincides with C. The secant AB then becomes the tangent HK.

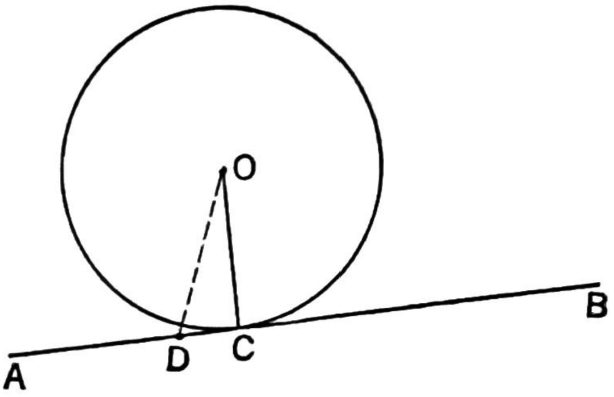
A *tangent* is an unlimited straight line that meets the circle at only one point. A tangent has therefore only one point in common with the circle. This point is called the *point of tangency*, or the *point of contact.* Thus HK is tangent to the ⊙ O at C. C is the point of contact.



TANGENTS

Proposition X. Theorem

159. A straight line perpendicular to a radius at its outer extremity is tangent to the circle.



Given OC a radius of ⊙ O, and $AB \perp OC$ at C.

Prove AB is tangent to ⊙ O.

Analysis. We must prove that C is the only point common to AB and the ⊙. Hence we must show that any point on AB other than C is not on the ⊙.

Proof. Let D be *any* point on AB other than C. Draw OD.

(1) $OC \perp AB$	(1) Given.
(2) $\therefore OD > OC$	(2) 118.
(3) $\therefore D$ is outside the ⊙.	(3) $OD >$ a radius of the ⊙.
(4) $\therefore$ all points on AB other than C are outside the ⊙.	(4) D is *any* point on AB other than C.
(5) $\therefore AB$ is tangent to the ⊙ at C.	(5) We have shown that C is the only point common to AB and the ⊙. (Def. of tangent.)

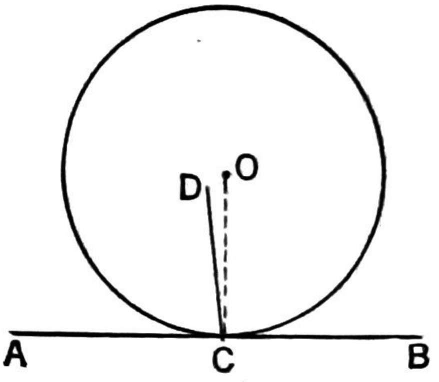
Q.E.D.

160. *Cor.* 1. A tangent to a circle is perpendicular to the radius drawn to the point of contact.

Outline of proof. All points on AB other than C are outside the ⊙. Hence OC is the shortest possible line from O to AB, i.e. it is $\perp AB$.

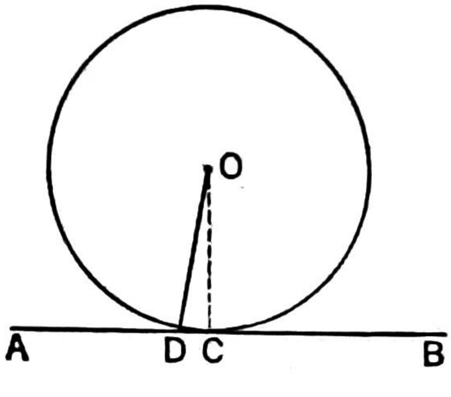
161. *Cor.* 2. A perpendicular to a tangent at the point of contact passes through the center of the circle.

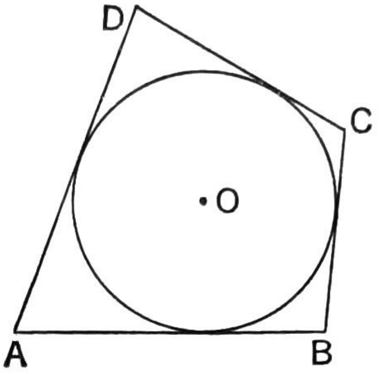
Outline of proof. Draw OC. Then $OC \perp AB$ at C, but $DC \perp AB$ at C. $\therefore DC$ coincides with OC, etc.



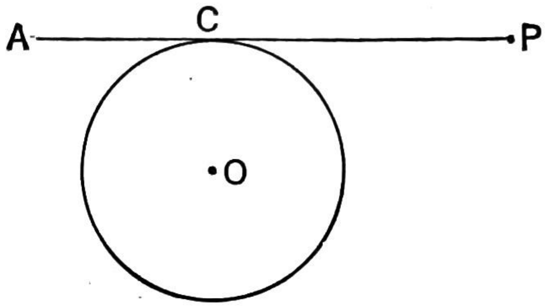
162. *Cor.* 3. A line from the center of a circle perpendicular to a tangent passes through the point of contact.

Outline of proof. Draw OC. Then $OC \perp AB$. But $OD \perp AB$. $\therefore OD$ coincides with OC, etc.





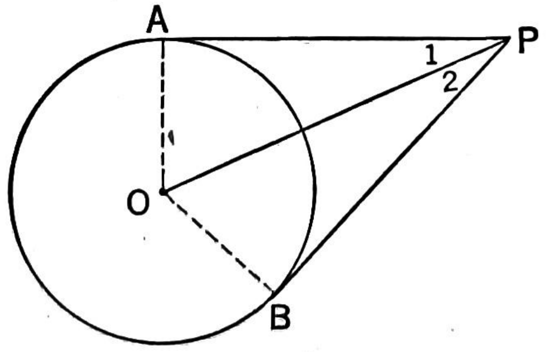
163. If the sides of a polygon are tangent to a circle, the polygon is said to be *circumscribed* about the circle, and the circle is *inscribed* in the polygon. Thus $ABCD$ is circumscribed about $\odot O$, and $\odot O$ is inscribed in $ABCD$.



164. If from an external point P a tangent is drawn to $\odot O$, the length of the tangent is the distance from P to the point of contact C. Briefly, we say that PC is the tangent from P to $\odot O$.

Proposition XI. Theorem

165. Tangents drawn to a circle from an external point are equal, and make equal angles with the line joining the point to the center of the circle.



Given $\odot O$, P an external point, PA and PB tangents to $\odot O$, PO connecting P with O.

Prove $PA = PB$, $\angle 1 = \angle 2$.

Plan is to get congruent $\triangle$s.

Proof. Draw OA and OB. Then

(1) $\angle A = 90°$, $\angle B = 90°$	(1) 160.
(2) $OP = OP$	(2) Iden.
(3) $OA = OB$	(3) Radii of the same ⊙ are equal.
(4) $\therefore$ rt. $\triangle OAP \cong$ rt. $\triangle OBP$	(4) 75.
(5) $\therefore PA = PB$ $\angle 1 = \angle 2$	(5) Corr. parts of congruent $\triangle$s are equal.

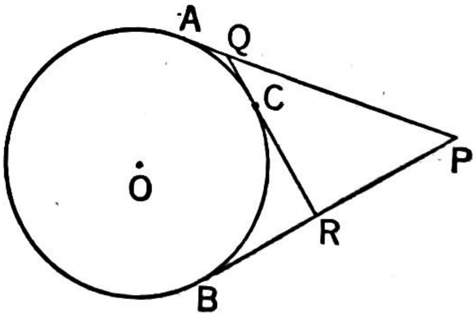
Q.E.D.

166. The line that joins the points of contact A and B is called the *chord of contact* of the two tangents.

Ex. 225. The line joining the center of a circle to the point of intersection of two tangents to the circle is the perpendicular bisector of the chord of contact.

Ex. 226. The sum of two opposite sides of a circumscribed quadrilateral is equal to the sum of the other two sides.

Ex. 227. If from an external point two tangents are drawn to a circle, and if from any point in the minor arc formed by the tangents another tangent is drawn to the circle, the three tangents form a triangle whose perimeter is constant.



Sug. Prove $PQ + QR + RP = PA + PB$.

167. The line joining the centers of two circles is called the *line of centers.* The segment between the centers is called the *centers-segment.*

168. Two circles are tangent to each other if both are tangent to the same line at the same point. When a line is tangent to two circles, it is called a *common tangent.* If the common tangent crosses the centers-segment, it is called a *common internal tangent;* if it does not, it is called a *common external tangent.* Thus, in Fig. 1, the Ⓢ are tangent externally at C, and AB is the common internal tangent. In Fig. 2, the Ⓢ are tangent internally at C, and AB is the common external tangent.

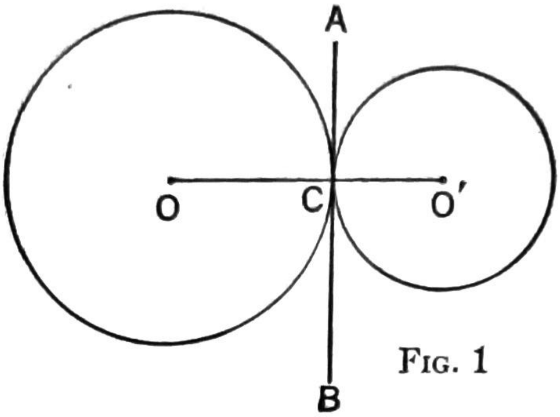


FIG. 1

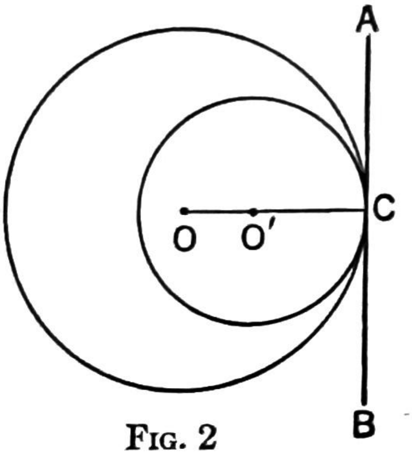
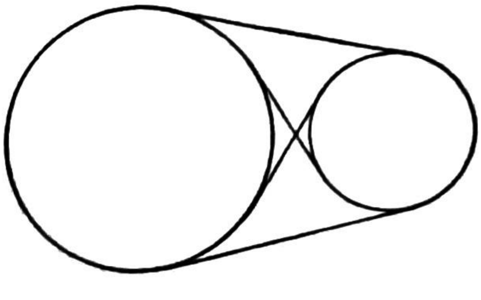


FIG. 2

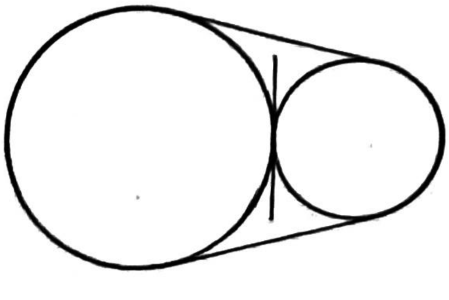
COMMON TANGENTS

169. The number of common tangents that can be drawn to two circles depends on the relative position of the circles.

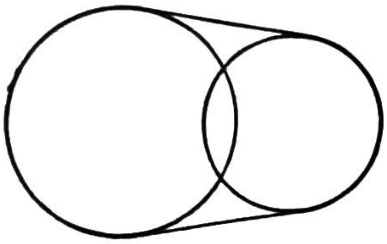
(*a*) When one circle is wholly outside of the other, two common external tangents and two common internal tangents can be drawn. The centers-segment is greater than the sum of the radii.



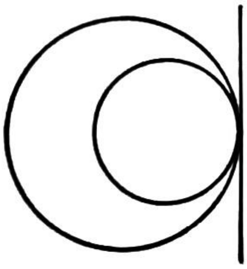
(*b*) When the circles are tangent externally, two common external tangents and one common internal tangent can be drawn. The centers-segment is equal to the sum of the radii.



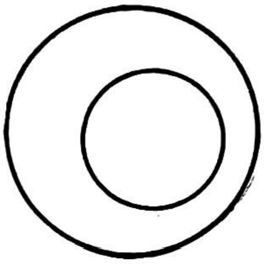
(*c*) When the circles intersect, two common external tangents can be drawn. The centers-segment is less than the sum but greater than the difference of the radii.



(*d*) When the circles are tangent internally, one common external tangent can be drawn. The centers-segment is equal to the difference of the radii.



(*e*) When one circle is wholly within the other, no common tangent can be drawn. The centers-segment is less than the difference of the radii, and if the circles are concentric (have a common center) the centers-segment is zero.



GROUP VIII

State whether the following statements are true or false, and give reasons.

(*a*) All tangents to a circle are equidistant from the center of the circle.

(*b*) A circle can always be drawn through three given points.

(*c*) A, B, C, D, taken in order, are four points on a circle. If chord AB = chord CD, then chord AC equals chord BD.

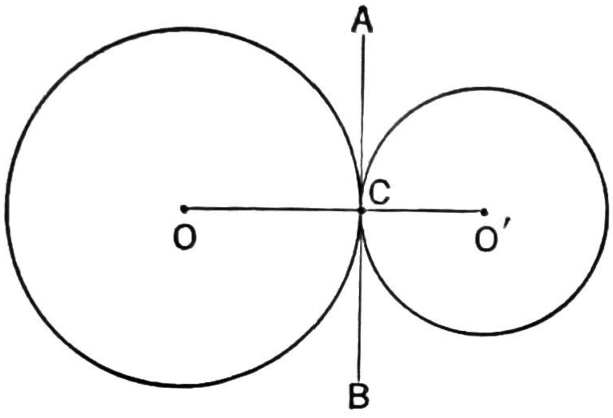
(*d*) AB is a diameter of $\odot O$, and chord $AC \parallel$ chord BD. Hence chord AC = chord BD.

(*e*) Quad. $ABCD$ is circumscribed about a $\odot$. If $AB = 5''$, $BC = 6''$, $CD = 7.5''$, then $DA = 6.5''$.

(*f*) If two circles are concentric, all lines tangent to the smaller circle and terminating in the larger are equal.

Proposition XII. Theorem

170. If two circles are tangent to each other, the line of centers passes through the point of contact.



Given ⓢ O and O' tangent to AB at C, and OO' the line of centers.

Prove OO' passes through C.

Analysis. If a perpendicular is drawn to AB *at* C, *it will pass through* O *and* O'. *Hence this perpendicular coincides with the line of centers.*

Proof. Draw a perpendicular to AB at C. Then

(1) The perpendicular will pass through O and O'	(1) 161.
(2) But the line of centers passes through O and O'	(2) Def. of line of centers.
(3) ∴ the line of centers coincides with the perpendicular.	(3) Two points determine a straight line.
(4) But the perpendicular passes through C.	(4) Const.
(5) ∴ the line of centers passes through C	(5) The line of centers coincides with the perpendicular. Q.E.D.

Note: The proof is the same if the circles are tangent internally.

Exercises

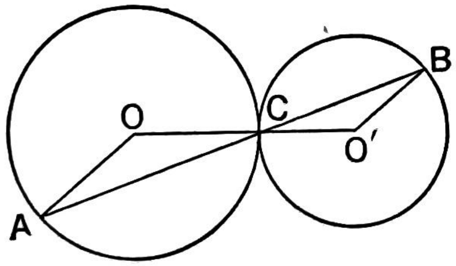
Ex. 228. Prove Prop. XII by drawing OC and $O'C$, and then showing that OCO' is a straight line.

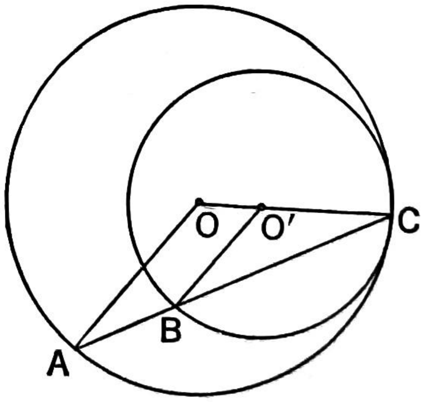
Ex. 229. The common external tangents of two circles are equal. Consider two cases: (*a*) when the circles are equal; (*b*) when the circles are unequal.

Ex. 230. The common internal tangents of two circles are equal.

Ex. 231. If two circles are tangent to each other, the tangents to them from any point of the common internal tangent are equal.

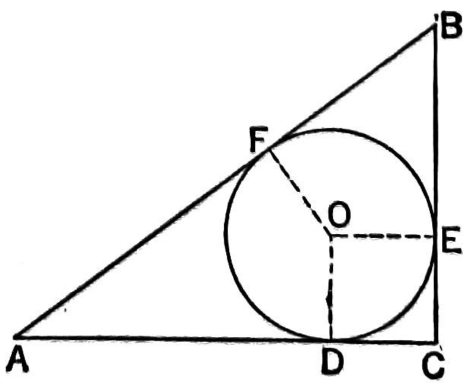
Ex. 232. If two circles are tangent externally, the common internal tangent bisects the common external tangents.



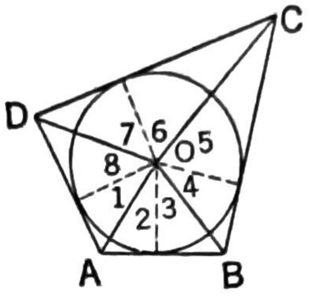


Ex. 233. In the annexed drawings ⓢ O and O' are tangent at C. Points A, B, and C are in a straight line. Prove $OA \parallel O'B$.

Ex. 234. The diameter of the circle inscribed in a right triangle is equal to the difference between the sum of the legs and the hypotenuse.



Ex. 235. The angles subtended at the center of a circle by any two opposite sides of a circumscribed quadrilateral are supplementary.

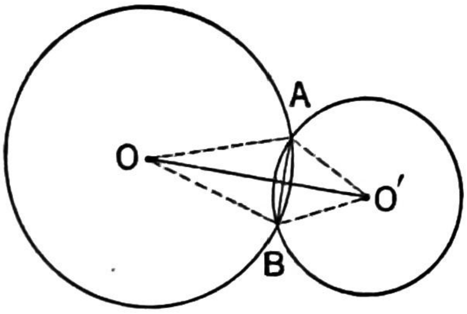


Sug. Prove ∠ 1 = ∠ 2, ∠ 3 = ∠ 4, ∠ 5 = ∠6, ∠ 7 = ∠ 8; then apply sum of ∠s about $O = 360°$.

INTERSECTING CIRCLES

Proposition XIII. Theorem

171. If two circles intersect, the line of centers is the perpendicular bisector of their common chord.



Given ⓢ O and O' intersecting at A and B.

Prove OO' is the perpendicular bisector of AB.

Plan is to get two points each equidistant from two other points.

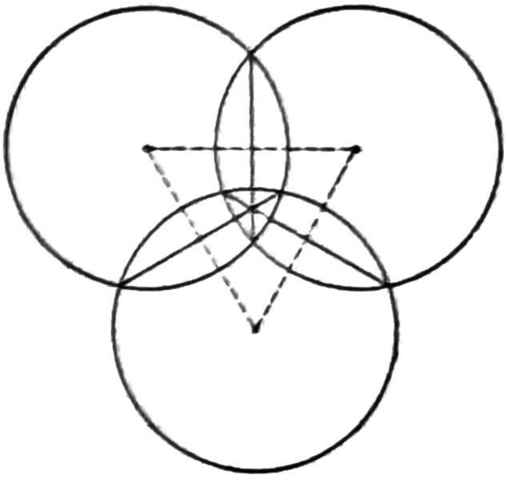
Proof. Draw OA, OB, $O'A$, $O'B$. Then

(1) O is equidistant from A and B	(1) Radii of the same ⊙ are equal.
(2) O' is equidistant from A and B	(2) Same reason.
(3) ∴ OO' is the ⊥ bisector of AB.	(3) 78.

Q.E.D.

Ex. 236. If three equal circles intersect in pairs, the three common chords pass through the same point.

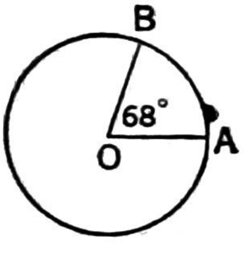
Sug. Draw the centers-segments, and then apply 122.

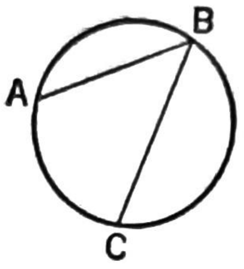


Ex. 237. If two circles intersect, the centers-segment is less than the sum but greater than the difference of the radii.

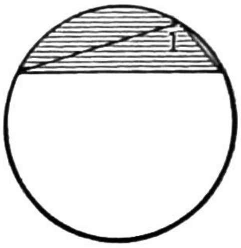
Measurement of Angles and Arcs

172. Imagine a circle divided into 360 equal parts, and the points of division connected with the center of the circle. We then have 360 equal angles at the center, since in the same circle equal arcs have equal central angles. If each little arc is called a degree of arc and each little central angle a degree of angle, then there are as many arc degrees in the circle as there are angle degrees at the center. It follows that if central $\angle AOB$ is 68°, then $\overset{\frown}{AB}$ is 68°. This does not mean that $\angle AOB$ equals $\overset{\frown}{AB}$. Arcs and angles are entirely unlike things. It simply means that 68 is the numerical measure of $\angle AOB$ and also of $\overset{\frown}{AB}$, or put in the form of an equation, the number of angle degrees in $\angle AOB$ = the number of arc degrees in $\overset{\frown}{AB}$. We shall assume that if the measure of a central angle is represented by any number, then the measure of its arc is represented by the same number. Briefly, we say, *A central angle is measured by its intercepted arc.*





173. An *inscribed angle* is an angle whose vertex is on the circle and whose sides are chords. Thus ∠ ABC is an inscribed angle, and $\overset{\frown}{AC}$ is its intercepted arc.



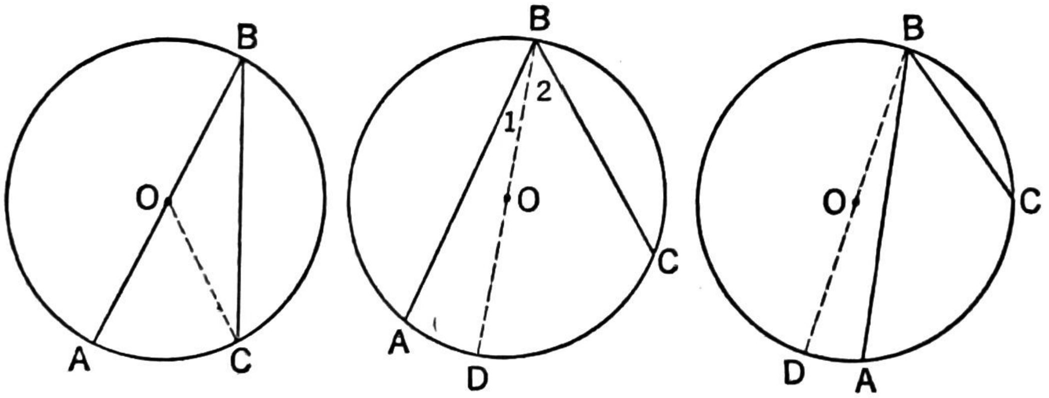
174. A *segment of a circle* is the space bounded by an arc and its chord. Thus the shaded portion is a segment.

175. An angle is *inscribed in a segment* if its vertex is on the arc of the segment and its sides terminate in the ends of the arc. Thus ∠ 1 is inscribed in the segment.

MEASUREMENT OF ANGLES

Proposition XIV. Theorem

176. An inscribed angle is measured by half its intercepted arc.



Given ∠ ABC inscribed in ⊙ O.

Prove ∠ ABC is measured by $\frac{1}{2}$ $\overset{\frown}{AC}$.

There are three possible cases to be considered: the center of the circle might be on one of the sides, or within the angle, or outside the angle.

Case I. When the center is on one of the sides of the angle.

Plan is to show that $\angle ABC$ *is half of central* $\angle AOC$, *and therefore* $\angle ABC$ *will have only half the measure of* $\angle AOC$.

Proof. Draw OC.

(1) $\angle AOC = \angle B + \angle C$	(1) 67.
(2) but $\angle B = \angle C$	(2) 42.
(3) $\therefore \angle AOC = 2 \angle B$	(3) Sub.
(4) $\therefore \angle B = \frac{1}{2} \angle AOC$	(4) Ax. 5.
(5) but $\angle AOC$ is measured by $\overset{\frown}{AC}$	(5) A central angle is measured by its intercepted arc.
(6) $\therefore \frac{1}{2} \angle AOC$ is measured by $\frac{1}{2} \overset{\frown}{AC}$	(6) Ax. 5.
(7) $\therefore \angle B$ is measured by $\frac{1}{2} \overset{\frown}{AC}$.	(7) Sub.

Case II. When the center is within the angle.

Plan is to make this case depend on Case I.

Proof. Draw the diameter BD. Then

(1) $\angle 1$ is measured by $\frac{1}{2} \overset{\frown}{AD}$	(1) Case I.
(2) $\angle 2$ is measured by $\frac{1}{2} \overset{\frown}{DC}$	(2) Case I.
(3) $\therefore (\angle 1 + \angle 2)$ is measured by $(\frac{1}{2} \overset{\frown}{AD} + \frac{1}{2} \overset{\frown}{DC})$	(3) Ax. 2.
(4) $\therefore (\angle 1 + \angle 2)$ is measured by $\frac{1}{2}(\overset{\frown}{AD} + \overset{\frown}{DC})$	(4) Factoring.
(5) $\therefore \angle ABC$ is measured by $\frac{1}{2} \overset{\frown}{AC}$	(5) Ax. 6.

Case III. When the center is outside the angle.

Plan is to make this case depend on Case I.

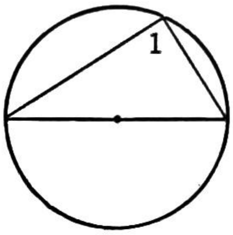
Proof. Draw the diameter BD. Then

(1) $\angle DBC$ is measured by $\frac{1}{2} \overset{\frown}{DC}$	(1) Case I.

(2) $\angle DBA$ is measured by $\frac{1}{2}\overset{\frown}{DA}$	(2) Case I.
(3) $\therefore (\angle DBC - \angle DBA)$ is measured by $(\frac{1}{2}\overset{\frown}{DC} - \frac{1}{2}\overset{\frown}{DA})$	(3) Ax. 3.
(4) $\therefore (\angle DBC - \angle DBA)$ is measured by $\frac{1}{2}(\overset{\frown}{DC} - \overset{\frown}{DA})$	(4) Factoring.
(5) $\therefore \angle ABC$ is measured by $\frac{1}{2}\overset{\frown}{AC}$	(5) Sub.

Q.E.D.

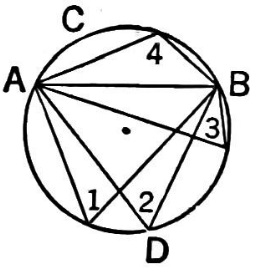
177. *Cor.* 1. An angle inscribed in a semicircle is a right angle.



178. *Cor.* 2. Angles inscribed in the same segment are equal.

Sug. Each $\angle$ is measured by half the same arc; e.g. $\angle 1 = \angle 2 = \angle 3$.

179. *Cor.* 3. An angle inscribed in a segment whose arc is greater than a semicircle is acute; if in less than a semicircle, it is obtuse.



Sug. $\angle 1$ is measured by $\frac{1}{2}\overset{\frown}{ACB}$, and $\overset{\frown}{ACB} < 180°$; $\angle 4$ is measured by $\frac{1}{2}\overset{\frown}{ADB}$, and $\overset{\frown}{ADB} > 180°$.

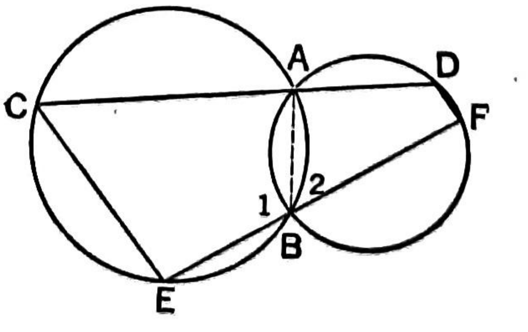
Exercises

Ex. 238. If a quadrilateral is inscribed in a circle its opposite angles are supplementary.

Ex. 239. A parallelogram inscribed in a circle is a rectangle.

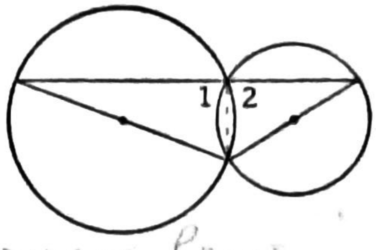
Ex. 240. A trapezoid inscribed in a circle is an isosceles trapezoid.

Ex. 241. If two circles intersect, and a line is drawn through each point of intersection terminated by the circles, the chords joining the ends of these lines are parallel.



Outline of proof. ∠ C = ∠ 2 because each is the supplement of ∠ 1. ∠ D is the supp. of ∠ 2. ∴ ∠ D is supp. of ∠ C, etc.

Ex. 242. Through one of the points of intersection of two circles a diameter of each circle is drawn. Prove that the line joining the ends of the diameters passes through the other point of intersection.



Sug. ∠ 1 = 90°, ∠ 2 = 90°, ∴ etc. INDIRECT PROOF

Ex. 243. An inscribed angle ABC is 30°. Prove that the chord AC is equal to a radius. PASSE OUTSIDE ARC

Ex. 244. Find the sum of three alternate angles of an inscribed hexagon.

Ex. 245. A circle constructed on one of the legs of an isosceles triangle as diameter bisects the base.

GROUP IX

State whether the following statements are true or false, and give reasons.

(*a*) If a central angle is doubled, its chord is also doubled.

(*b*) If two circles are so situated that the centers-segment is greater than the sum of their radii, then only two common tangents can be drawn to these circles.

(*c*) AB and CD are two chords of a circle, and they intersect at E. If $CE = BE$, then $\widehat{AC} = \widehat{BD}$.

(*d*) If an angle inscribed in a circle is 40°, then its intercepted arc is 20°.

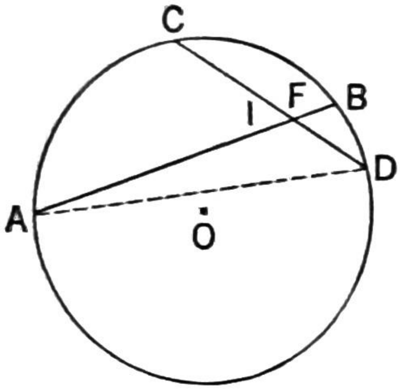
(*e*) If a circle of 5″ radius circumscribes a right triangle, the hypotenuse is 10″.

(*f*) $\widehat{AB}$ of ⊙ O is 120°. Hence any angle inscribed in the segment whose arc is AB must be 120°.

(*g*) If two concentric ⊙ have radii of 5″ and 2″ respectively, it is possible for a ⊙ whose diameter is 7″ to be tangent to both ⊙.

Proposition XV. Theorem

180. An angle formed by two chords intersecting within a circle is measured by half the sum of its intercepted arc and that of its vertical angle.



Given ⊙ O with chords AB and CD intersecting at F within the ⊙, thus forming ∠ 1.

Prove ∠ 1 is measured by $\frac{1}{2}(\overset{\frown}{CA} + \overset{\frown}{BD})$.

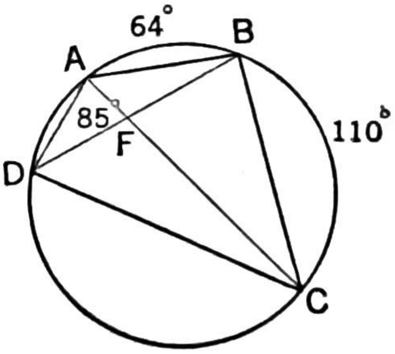
Plan is to get ∠ 1 *equal to the sum of two inscribed angles.*

Proof. Draw AD, thus forming △ AFD. Then

(1) $\angle 1 = \angle D + \angle A$	(1) 67.
(2) but $\angle D$ is measured by $\frac{1}{2}\overset{\frown}{CA}$ and $\angle A$ is measured by $\frac{1}{2}\overset{\frown}{BD}$	(2) 176.
(3) ∴ ∠ 1 is measured by $(\frac{1}{2}\overset{\frown}{CA} + \frac{1}{2}\overset{\frown}{BD})$	(3) Sub.
(4) ∴ ∠ 1 is measured by $\frac{1}{2}(\overset{\frown}{CA} + \overset{\frown}{BD})$	(4) Factoring.

Q.E.D.

Ex. 246. $ABCD$ is an inscribed quadrilateral. $\overset{\frown}{AB} = 64°$, $\overset{\frown}{BC} = 110°$; $\angle AFD$, formed by the diagonals, is 85°. Find the number of degrees in each interior angle of the quadrilateral.



Solution. $85° = \frac{1}{2}(\overset{\frown}{AD} + 110°)$

∴ $\overset{\frown}{AD} = 60°$

∴ $\overset{\frown}{DC} = 126°$

$\therefore \angle DAB = \frac{1}{2}(110^\circ + 126^\circ) = 118^\circ$, etc.

Ex. 247. If two perpendicular chords intersect within a circle, the sum of a pair of opposite intercepted arcs is equal to a semicircle.

Ex. 248. If a circle is drawn upon the radius of another circle as diameter, any chord of the greater circle passing through the point of contact of the circles is bisected by the smaller circle.

Ex. 249. All equal chords of a circle are tangent to an inner concentric circle.

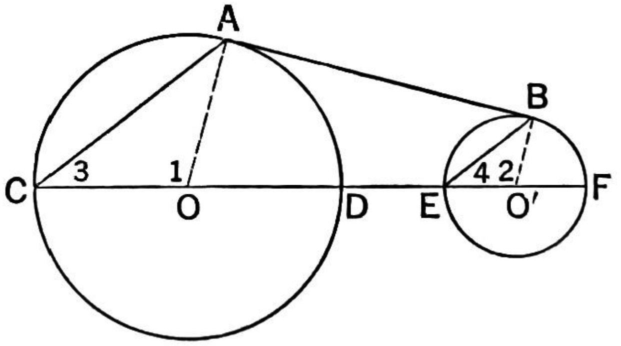
Ex. 250. If from the extremities of a diameter perpendiculars are drawn upon any chord (produced, if necessary), the feet of the perpendiculars are equidistant from the center of the circle.

Ex. 251. If two secants to a circle from an external point are equal, the intercepted chords are equal.

Ex. 252. The radius of the circle circumscribed about an equilateral triangle is two thirds of the altitude of the triangle.

Sug. Apply 124.

Ex. 253. AB is a common external tangent to circles O and O'. The line of centers cuts one circle in C and D and the other in E and F. Prove $AC \parallel BE$.



Sug. Draw OA and $O'B$. Prove $\angle 1 = \angle 2$, then prove $\angle 3 = \angle 4$.

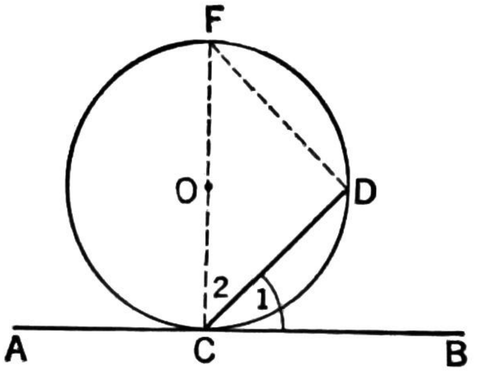
If in this figure chords AD and BF are drawn, prove $AD \parallel BF$.

If $\angle 3$ is 40°, how many degrees are there in $\angle 2$? in $\overset{\frown}{AD}$? in $\overset{\frown}{BF}$?

Why must the common tangent AB be less than the centers-segment OO'?

Proposition XVI. Theorem

181. An angle formed by a tangent and a chord drawn from the point of contact is measured by half the intercepted arc.



Given AB tangent to $\odot O$ at C, and chord CD forming $\angle 1$.

Prove $\angle 1$ is measured by $\frac{1}{2}\overset{\frown}{CD}$.

Plan is to get an inscribed angle that intercepts arc CD and to prove that this angle equals $\angle 1$.

Proof. Draw the diameter CF and then draw FD.

(1) $CF \perp AB$	(1) 160.
(2) $\therefore \angle 1$ is the complement of $\angle 2$	(2) Def. of complementary angles.
(3) $\angle D = 90°$	(3) 177.
(4) $\therefore \angle F$ is the complement of $\angle 2$	(4) 66.
(5) $\therefore \angle 1 = \angle F$	(5) Complements of the same angle are equal.
(6) But $\angle F$ is measured by $\frac{1}{2}\overset{\frown}{CD}$	(6) 176.
(7) $\therefore \angle 1$ is measured by $\frac{1}{2}\overset{\frown}{CD}$	(7) Sub.

Q.E.D.

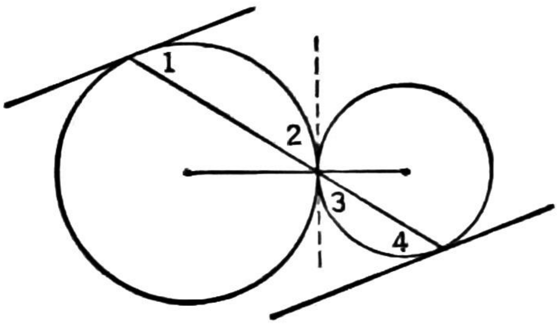
Exercises

Ex. 254. The angle formed by a tangent and a chord to the point of contact is 38°. How many degrees are there in the arcs of this chord?

Ex. 255. A chord is parallel to the tangent drawn through the midpoint of the subtended arc.

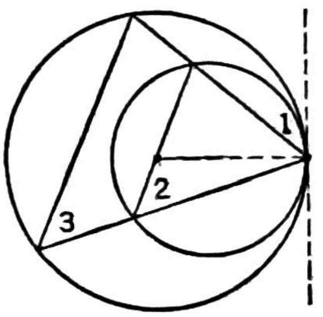
Ex. 256. If two circles are tangent externally and a line is drawn through the point of contact terminated by the circles, the tangents at its ends are parallel.

Sug. Draw the common internal tangent. Then $\angle 1 = \angle 2 = \angle 3 = \angle 4$, etc.

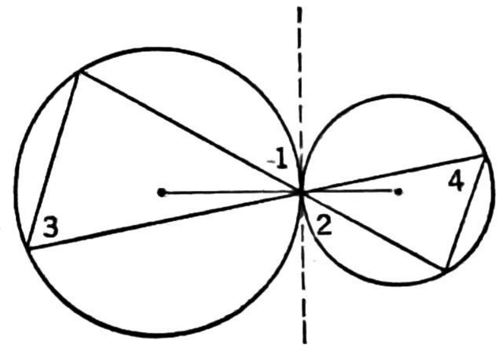


Ex. 257. If two circles are tangent internally or externally, and two lines are drawn through the point of contact terminated by the circles, the chords joining the ends of these lines are parallel.

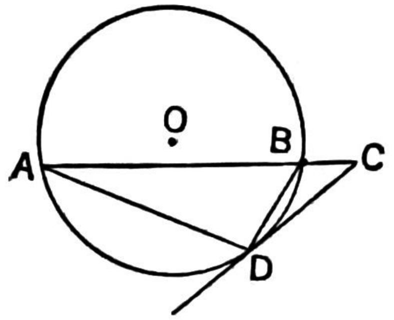
Sug. Draw the common tangent and then prove $\angle 1 = \angle 2 = \angle 3$.



Here prove $\angle 3 = \angle 1 = \angle 2 = \angle 4$

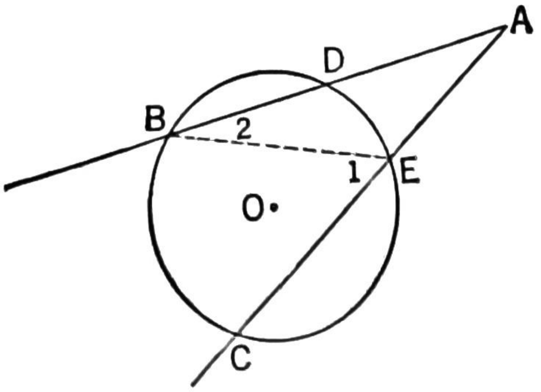


Ex. 258. In the annexed figure CD is tangent to $\odot O$ at D. $\angle C = 42°$, $\widehat{BD} = 32°$. Find in degrees the value of each angle of $\triangle ABD$.



Proposition XVIIa. Theorem

182. An angle formed by two secants drawn from an external point to a circle is measured by half the difference of the intercepted arcs.



Given $\odot O$, $\angle A$ formed by secants AB and AC drawn from A.

Prove $\angle A$ is measured by $\frac{1}{2}(\overset{\frown}{BC} - \overset{\frown}{DE})$.

Plan is to make $\angle A$ *equal the difference of two inscribed angles.*

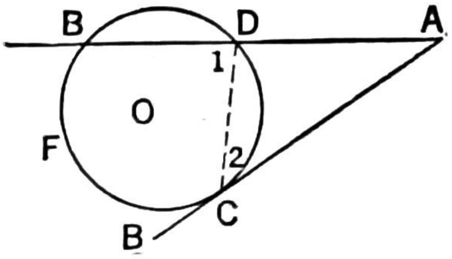
Proof. Draw BE. Then

(1) $\angle 1 = \angle 2 + \angle A$	(1) 67.
(2) $\therefore \angle A = \angle 1 - \angle 2$	(2) Ax. 3.
(3) but $\angle 1$ is measured by $\frac{1}{2}\overset{\frown}{BC}$ and $\angle 2$ is measured by $\frac{1}{2}\overset{\frown}{DE}$	(3) 176.
(4) $\therefore \angle A$ is measured by $(\frac{1}{2}\overset{\frown}{BC} - \frac{1}{2}\overset{\frown}{DE})$	(4) Sub.
(5) $\therefore \angle A$ is measured by $\frac{1}{2}(\overset{\frown}{BC} - \overset{\frown}{DE})$	(5) Factoring.

Q.E.D.

Proposition XVIIb. Theorem

183. An angle formed by a secant and a tangent drawn from an external point to a circle is measured by half the difference of the intercepted arcs.

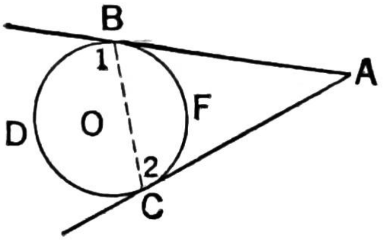


Prove $\angle A$ is measured by $\frac{1}{2}(\widehat{BFC} - \widehat{DC})$.

Sug. Draw CD. Then $\angle 1 = \angle 2 + \angle A$, etc.

Proposition XVIIc. Theorem

184. An angle formed by two tangents drawn from an external point to a circle is measured by half the difference of the intercepted arcs.



Prove $\angle A$ is measured by $\frac{1}{2}(\widehat{BDC} - \widehat{BFC})$.

Sug. Draw BC. Then $\angle 1 = \angle 2 + \angle A$, etc.

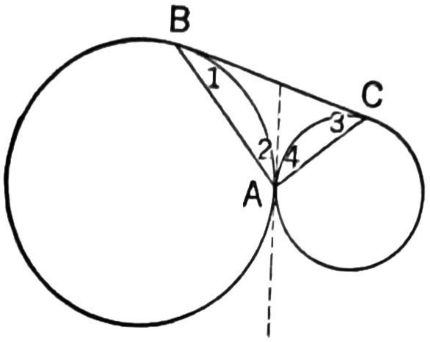
Exercises

Ex. 259. An angle inscribed in a circle and an angle formed by two tangents drawn from an external point to the circle intercept the same arc. The inscribed angle is 53°. Find the angle formed by the tangents.

Ex. 260. A triangle ABC is inscribed in a circle. $\angle B$ is 28°. Minor arc AB is 110°. What angle does a tangent at A make with BC produced to meet the tangent?

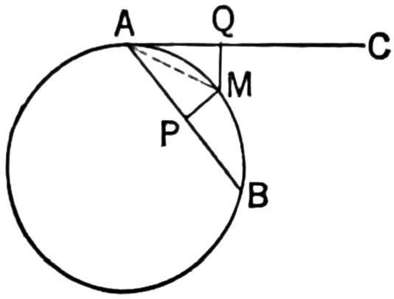
Ex. 261. Three angles whose sides are parallel each to each are formed respectively by two radii, two chords, and two tangents of a circle. The angle formed by the tangents is 80°. Find the number of degrees in the arcs intercepted by the sides of each angle.

Ex. 262. Two equal chords are produced till they meet. The angle formed by extending the chords is 20°, and the smaller arc intercepted by them is $\frac{1}{15}$ of the circle. Find each angle of the quadrilateral formed by joining the ends of the chords.



Ex. 263. Two circles are tangent externally at *A*, and a common external tangent touches them at *B* and *C* respectively. Prove ∠ *BAC* is a right angle.

Sug. Draw the common internal tangent. Then ∠ 1 = ∠ 2, ∠ 3 = ∠ 4, but ∠ 1 + ∠ 2 + ∠ 4 + ∠ 3 = 180°, etc.



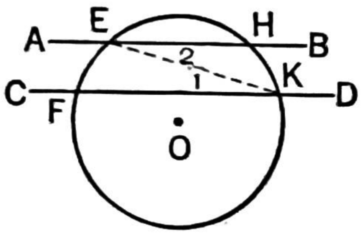
Ex. 264. If, from any point on a circle, a chord and a tangent are drawn, the perpendiculars dropped on them from the midpoint of the intercepted arc are equal.

Sug. Draw *AM* and prove that it bisects ∠ *BAC*.

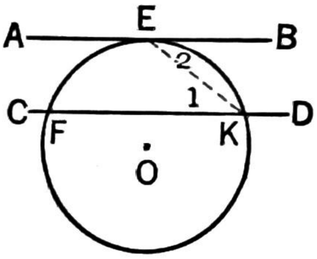
PARALLEL CHORDS AND TANGENTS

Proposition XVIII. Theorem

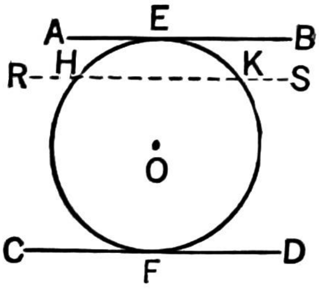
185. Arcs of a circle intercepted by two parallel lines are equal.



Case I



Case II



Case III

There are three possible cases to be considered: both parallels may be secants, one may be a secant and the other a tangent, or both may be tangents.

Case I. *Given* $\odot O$ with the $\parallel$ secants AB and CD intercepting the arcs EF and HK.

Prove $\overset{\frown}{EF} = \overset{\frown}{HK}$.

Plan is to make these arcs the arcs of two equal inscribed angles.

Proof. Draw EK. Then

(1) $\angle 1 = \angle 2$	(1) 56.
(2) But $\angle 1$ is measured by $\frac{1}{2}\overset{\frown}{EF}$ and $\angle 2$ is measured by $\frac{1}{2}\overset{\frown}{HK}$	(2) 176.
(3) $\therefore \frac{1}{2}\overset{\frown}{EF} = \frac{1}{2}\overset{\frown}{HK}$	(3) Sub.
(4) $\therefore \overset{\frown}{EF} = \overset{\frown}{HK}$	(4) Ax. 4.

Case II. *Given* $\odot O$ with the tangent $AB \parallel$ secant CD, and intercepting the arcs EF and EK.

Prove $\overset{\frown}{EF} = \overset{\frown}{EK}$.

Plan is to make these arcs the arcs of two equal angles.

Proof. Draw EK. Then

(1) $\angle 1 = \angle 2$	(1) 56.
(2) But $\angle 1$ is measured by $\frac{1}{2}\overset{\frown}{EF}$	(2) 176.
(3) And $\angle 2$ is measured by $\frac{1}{2}\overset{\frown}{EK}$	(3) 181.
(4) $\therefore \frac{1}{2}\overset{\frown}{EF} = \frac{1}{2}\overset{\frown}{EK}$	(4) Sub.
(5) $\therefore \overset{\frown}{EF} = \overset{\frown}{EK}$	(5) Ax. 4.

Case III. *Given* $\odot O$ with the $\parallel$ tangents AB and CD intercepting arcs EHF and EKF.

Prove $\overset{\frown}{EHF} = \overset{\frown}{EKF}$.

Plan is to apply Case II.

Proof. Draw secant $RS \parallel AB$, cutting the $\odot$ in H and K.

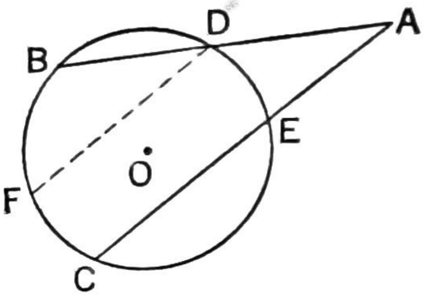
Then

(1) $RS \parallel CD$	(1) 61.
(2) $\overset{\frown}{EH} = \overset{\frown}{EK}$	(2) Case II.
(3) $\overset{\frown}{HF} = \overset{\frown}{KF}$	(3) Case II.
(4) $\therefore \overset{\frown}{EHF} = \overset{\frown}{EKF}$	(4) Ax. 2.

Q.E.D.

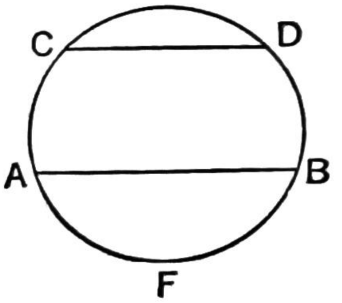
Ex. 265. Prove Case III by drawing OE and OF, proving that EOF is a straight line, and then using the alternate interior angles AEF and EFD.

Ex. 266. Prove Prop. XVIIa by drawing $DF \parallel AC$.



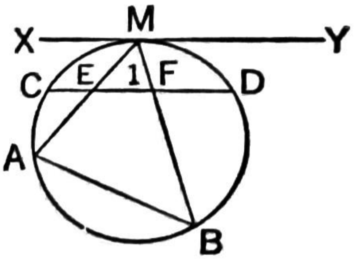
Ex. 267. In the annexed figure $CD \parallel AB$, $\overset{\frown}{AC} = 65°$, $\overset{\frown}{CD} = \frac{2}{3}\overset{\frown}{AFB}$. Find the number of degrees in $\overset{\frown}{CD}$ and in $\overset{\frown}{AFB}$.

Sug. Let $\overset{\frown}{AFB} = 3x$ and $\overset{\frown}{CD} = 2x$.



Ex. 268. A, B, C are three points on a circle. The bisector of $\angle ABC$ meets the circle again at D. DE is drawn $\parallel AB$ and meets the circle again at E. Prove $DE = BC$.

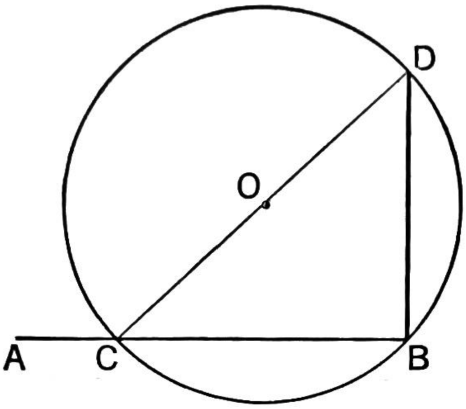
Ex. 269. If chord $CD \parallel$ tangent XY prove that $\angle A = \angle 1$.



CONSTRUCTIONS

Proposition XIX. Problem

186. To draw a perpendicular to a given line at one of its end points, without prolonging the line.



Given line AB.

Required to draw a perpendicular to AB at B, without prolonging the line AB.

Plan is to get an angle at B which will be inscribed in a semicircle.

Construction. (*a*) Take any point O (not in the required perpendicular, and somewhere between A and B), and with OB as a radius draw an arc cutting AB in B and C.

(*b*) Draw CO and prolong it to meet the arc at D.

(*c*) Draw BD. Then BD is the required perpendicular.

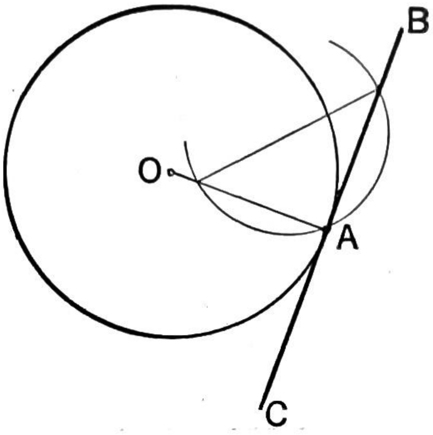
Proof. $\angle B$ is a right angle (by 177).

Remark: This method can be used to draw a perpendicular to a given line at any point of the line.

Query: Can this problem be solved if point O is taken somewhere above AB and to the left of AB? to the right of AB?

Proposition XXa. Problem

187. To draw a tangent to a circle at a point on the circle.



Given ⊙ O and point A on it.

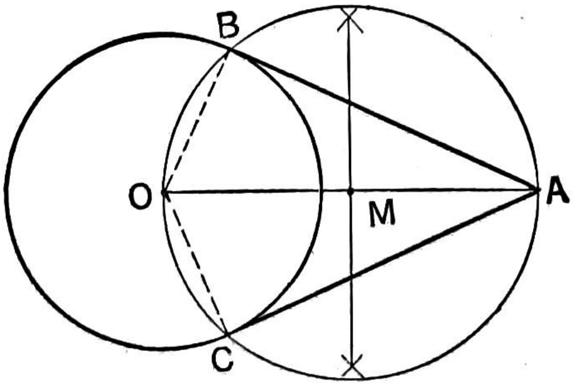
Required to draw a tangent to ⊙ O at A.

Analysis. If such a tangent were drawn, it would be $\perp OA$ *at* A. *Hence we must draw a perpendicular to* OA *at* A.

Construction. Draw OA, and at A draw $BC \perp OA$ (by the method of 186). Then BC is the required tangent (159).

Proposition XXb. Problem

188. To draw a tangent to a circle from a point outside the circle.



Given ⊙ O and point A outside the ⊙.

Required to draw a tangent to the ⊙ from A.

Plan is to draw a line from A which shall be ⊥ a radius at its outer extremity.

Construction. (*a*) Draw OA and find its midpoint M.

(*b*) With M as center, and MO as radius, draw a ⊙ cutting the given ⊙ in B and C.

(*c*) Draw AB and AC. Then AB and AC are the required tangents.

Proof. Draw OB and OC. Then $\angle B = 90°$, $\angle C = 90°$ (by 177).

∴ AB and AC are tangents to the ⊙ (by 159).

Q.E.D.

Note that from a point outside a ⊙ it is always possible to draw two tangents to the ⊙.

GROUP X

State whether the following statements are true or false, and give reasons.

(*a*) A circle can be circumscribed about a rectangle.

(*b*) If an angle is inscribed in a segment whose arc is 160°, the angle is 100°.

(*c*) The bisectors of the angles of a circumscribed polygon meet in a point.

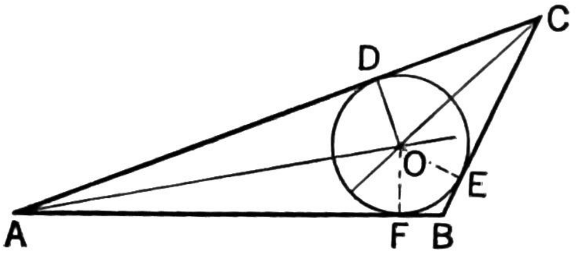
(*d*) The bisectors of all angles inscribed in a given segment of a circle meet in a point.

(*e*) If two Ⓢ intersect, their common chord can never be the perpendicular bisector of their centers-segment.

(*f*) AB is a chord of ⊙ O, and AC is a tangent to the same circle. If $\angle BAC$ is 42°, then any angle inscribed in the segment whose arc is AB must be 138°.

Proposition XXI. Problem

189. To inscribe a circle in a given triangle.



Given △ ABC.

Required to inscribe a ⊙ in △ ABC.

Analysis. If the ⊙ *were drawn, its center would be equidistant from the sides of the* △. *Hence we must locate a point that is equidistant from the sides of the* △. 121 *suggests the method.*

Construction. (*a*) Bisect $\angle A$ and $\angle C$, and call O the point where the bisectors meet.

(*b*) Draw $OD \perp AC$.

(*c*) With O as center, and OD as radius, draw a ⊙. Then this will be the required ⊙.

Proof. Draw $OE \perp BC$, $OF \perp AB$.

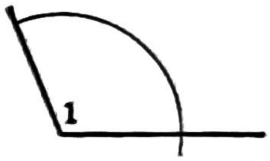
(1) $OD = OE = OF$	(1) 121.
(2) ∴ the ⊙ drawn with O as center and OD as radius must pass through D, E, and F	(2) Radii of the same ⊙ are equal.
(3) ∴ AB, BC, AC are tangents to this ⊙	(3) 159.
(4) ∴ the ⊙ is inscribed in △ ABC.	(4) 163.

Q.E.D.

Note: The point O is called the *incenter* of △ ABC.

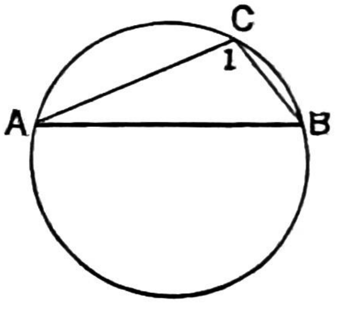
Proposition XXII. Problem

190. Upon a given line as chord, to draw a segment of a circle such that any angle inscribed in it shall equal a given angle.

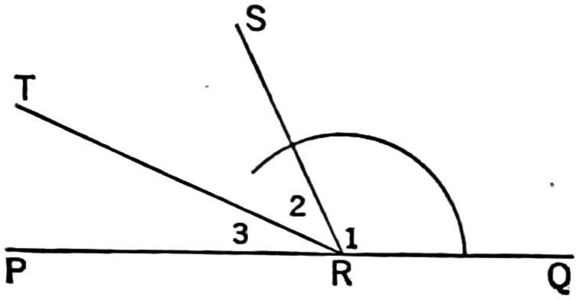


Given line AB and ∠ 1.

Required to draw, upon AB as chord, a segment of a circle such that any angle inscribed in it shall equal ∠ 1.



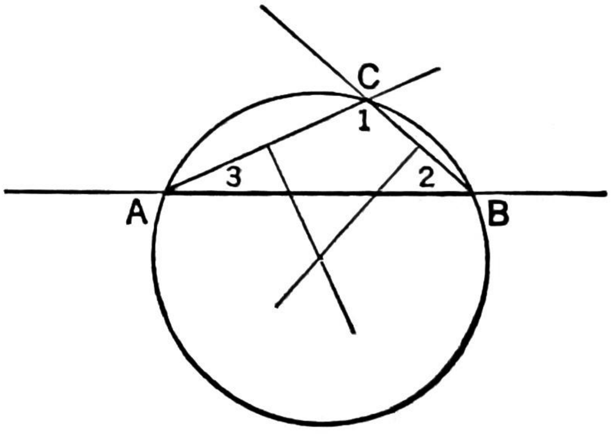
Analysis. Let us assume the problem done. Then AB is the base of a △ ABC, ∠ 1 is opposite the base, and the ⊙ circumscribes △ ABC. Reversing our steps, this suggests that we draw a △ with AB as base and ∠ 1 as angle opposite, and then circumscribe a ⊙ about this △.



Construction. (*a*) On any line PQ take a point R, and at R construct an angle equal to ∠ 1.

(*b*) Divide the ∠ PRS into any two parts, such as ∠ 2 and ∠ 3, by the line RT. Then ∡ 1, 2, and 3 can be made the angles of a △, since their sum is a straight angle.

(*c*) Measure off a length equal to AB.



(*d*) At B construct an angle equal to $\angle 2$.

(*e*) At A construct an angle equal to $\angle 3$, and prolong the lines till they meet at C. We now have a $\triangle ABC$ with AB as base and $\angle 1$ as angle opposite the base.

(*f*) Circumscribe a $\odot$ about $\triangle ABC$.

Then segment ABC is the required segment.

Proof. It has AB as chord; and $\angle C$, inscribed in the segment, equals $\angle 1$ [because $\angle C = 180° - (\angle 2 + \angle 3)$, $\angle 1 = 180° - (\angle 2 + \angle 3)$]. Furthermore, any angle inscribed in segment ACB must equal $\angle 1$ (by 178).

Q.E.D.

Proposition XXIII. Problem

191. To construct a triangle, given two sides and the angle opposite one of them. (a, b, $\angle A$.)

There are various possibilities, depending upon the size of $\angle A$ and the relative lengths of a and b.

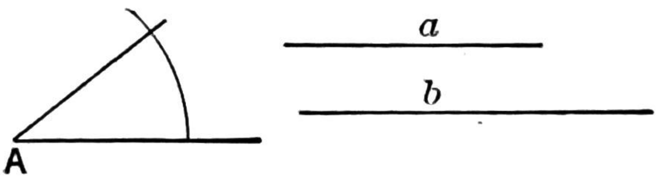
1. *When $\angle A$ is acute.*

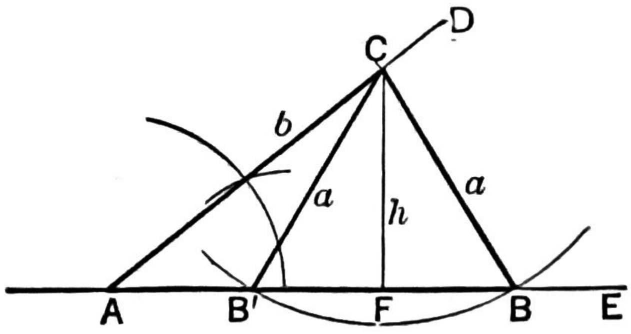
(*a*) Construct $\angle DAE$ equal to $\angle A$.

(*b*) On AD take AC equal to b.

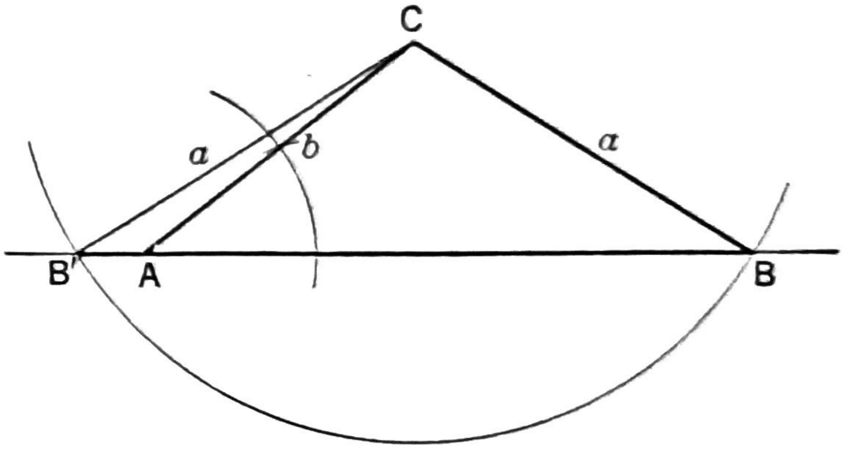
(*c*) With C as center, and a as radius, draw an arc cutting AE in B and B'.

(*d*) Draw BC and $B'C$. Then the $\triangle$ ABC and $AB'C$ fulfill the requirements, and we have two distinct solutions.



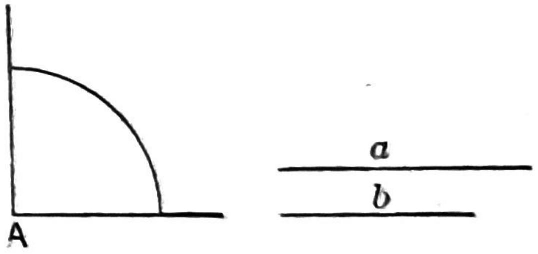


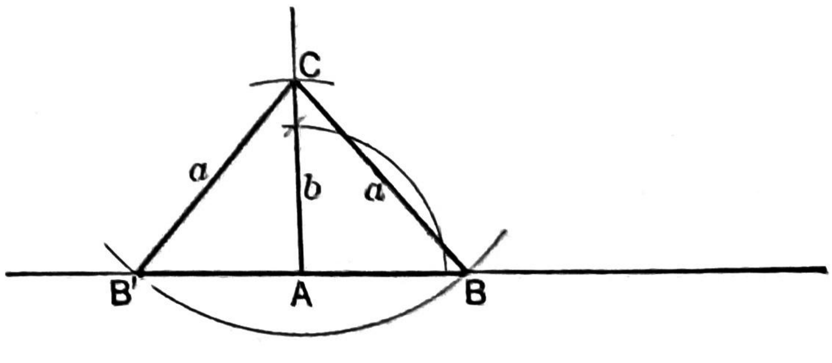
Discussion. In the above drawing, a was given $< b$ but $>$ the perpendicular distance from C to AB (denoted by h). If $a = h$, there is one solution — the right triangle ACF. If $a < h$, there is no solution. If $a = b$, the arc drawn with C as center and with a as radius will cut the line AE in B and A, and we then have one solution — the isosceles $\triangle ABC$. If $a > b$, the arc drawn will cut the line in two points, B and B'. $\triangle ABC$ fulfills the requirements, but $\triangle AB'C$ does not, since it does not contain the given acute $\angle$. We then have only one solution.



2. *When* $\angle$ *A is right.*

Here the arc cuts the line in points B and B', and we have two congruent right $\triangle$ ABC and $AB'C$ which fulfill the requirements.

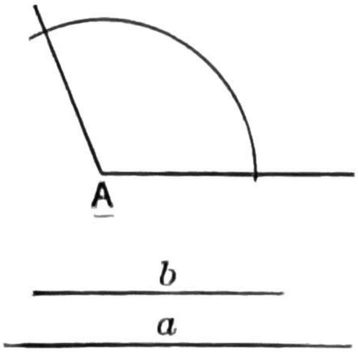


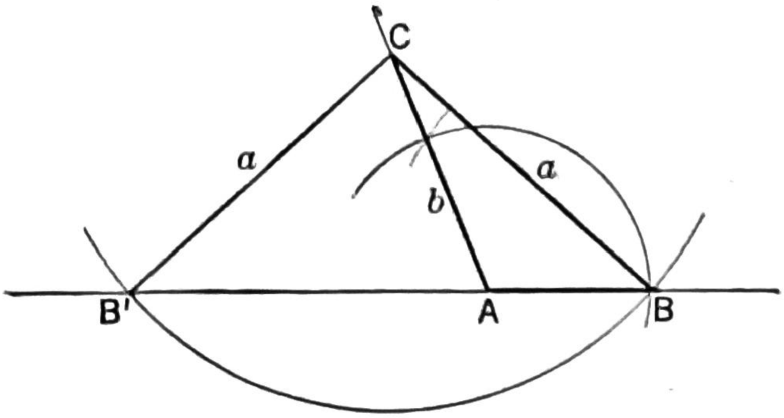


Discussion. In the drawing, we took $a > b$. It cannot be anything else, since the hypotenuse of a right △ is the longest side. Hence if $a < b$ there is no solution, and if $a = b$ there is no solution.

3. *When ∠ A is obtuse.*

Here the arc cuts the line in B and B'. △ ABC fulfills the requirements, but △ $AB'C$ does not, because it does not contain the given obtuse ∠ A. There is therefore only one solution.





Discussion. In the drawing, we took $a > b$. It cannot be anything else, since ∠ A is obtuse and is therefore the largest angle of the △. ∴ a must be the largest side. Hence if $a < b$ there is no solution, and if $a = b$ there is no solution. We may therefore completely summarize our results, as follows:

When a, b, and ∠ A are given,

$$\angle A \text{ acute} \left\{ \begin{array}{l} a < b \text{ but } a > h\text{, two solutions} \\ a < b \text{ but } a = h\text{, one sol. (a right } \triangle) \\ a < b \text{ and } a < h\text{, no sol.} \\ a = b\text{, one solution (an isos. } \triangle) \\ a > b\text{, one solution} \end{array} \right.$$

$$\angle A \text{ right} \begin{cases} a < b, \text{ no solution} \\ a = b, \text{ no solution} \\ a > b, \text{ one distinct solution (2 congruent right } \triangle s) \end{cases}$$

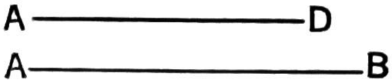
$$\angle A \text{ obtuse} \begin{cases} a < b, \text{ no solution} \\ a = b, \text{ no solution} \\ a > b, \text{ one solution} \end{cases}$$

Note that the only time there can be two distinct solutions of this problem is when $\angle A$ is acute and $b > a > h$.

SOLUTION OF CONSTRUCTION PROBLEMS

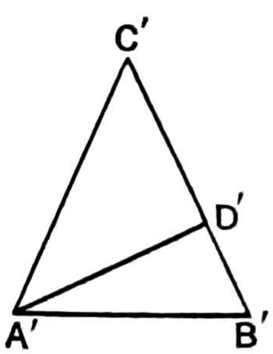
192. Construction problems often present considerable difficulty, and just how to proceed is far from obvious. The pupil should study carefully the following examples, and pay special attention to the method of attack.

Ex. 270. Construct an isosceles triangle having given the base and the altitude to one of the equal sides.

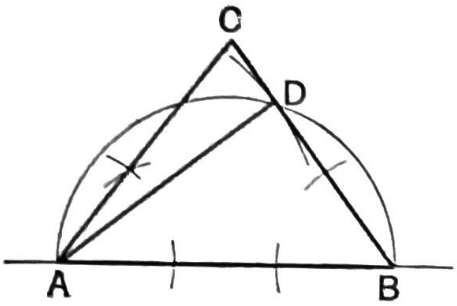


Given AB the base and AD the altitude to one of the legs of an isosceles $\triangle$.

Required to construct the triangle.



Analysis. Assume the problem done, and mark heavy the given parts. From a study of the figure we see that $\triangle A'D'B'$ *is fully determined. We therefore make* $\triangle A'D'B'$ *the basis of the construction. Since the* $\triangle A'B'C'$ *is isosceles,* $\angle C'A'B' = \angle B'$. *Having constructed* $\triangle A'D'B'$, *we must construct at* A' *an angle equal to* $\angle B'$. *We thus locate point* C', *and the required* $\triangle$ *is completely determined.*

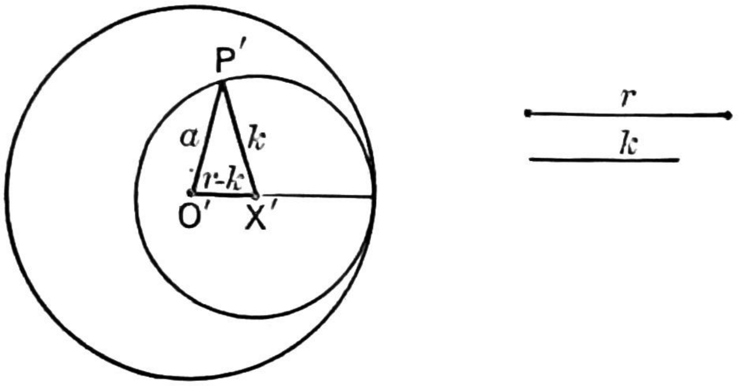


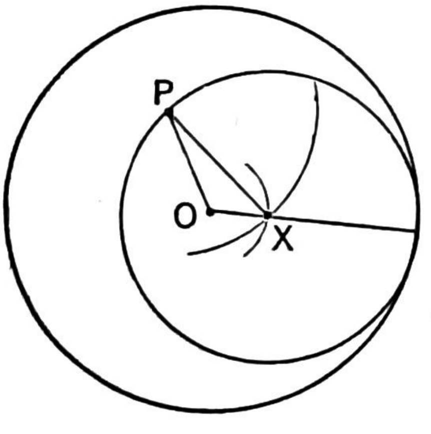
Construction. On an indefinite line lay off AB, and with AB as diameter draw a semicircle. With A as center and AD as radius draw an arc cutting the semicircle at D. At A construct $\angle CAB$ equal to $\angle B$. Prolong the lines till they meet at C. Then $\triangle ABC$ is the required $\triangle$.

Proof. $\angle ADB$ is a right angle (177). $\therefore AD \perp BC$; also $AC = BC$ (73). $\therefore \triangle ABC$ is isosceles with AB as the base and AD the altitude upon one of the legs.

Discussion. In order that a construction shall be possible, AB must be $> AD$.

Ex. 271. Construct a circle with given radius k, which will be tangent to a given circle of radius r, and pass through a given point P within the circle.





Given $\odot O$ of radius r, P a given point within the $\odot$, k a given length.

Required to draw a $\odot$ of radius k, which will be tangent to $\odot O$ and pass through P.

Analysis. Assume the problem done. The Ⓢ *are tangent internally, and* *therefore the centers-segment equals* $r - k$. $\triangle O'P'X'$ *is determined, because its three sides are known. This locates* X', *the center of the required* ⊙.

Construction. Draw OP. With P as center and k as radius draw an arc. With O as center and $r - k$ as radius draw an arc cutting the former arc at X. With X as center and k as radius draw a ⊙. This is the required ⊙.

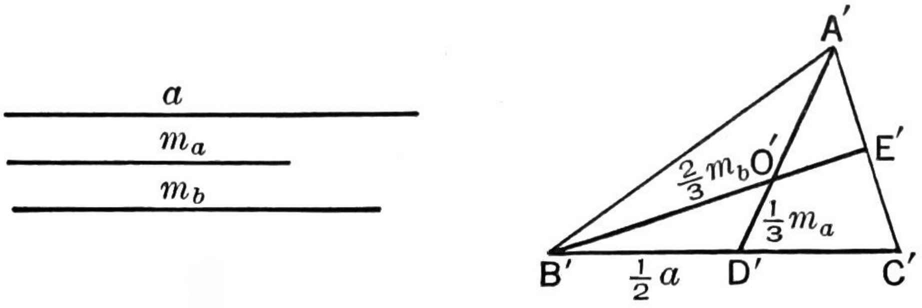
Proof. The centers-segment equals the difference of the radii $(r - k)$. ∴ the ⓢ are tangent internally. $XP = k$. ∴ the ⊙ has the given radius k and passes through the given point P.

Discussion. Denote the distance OP by d. Then since the sum of any two sides of a △ must be > the third side, we have $d + k > r - k$, $d + r - k > k$, $k + r - k > d$, or simplifying we get $d > r - 2\,k$, $d > 2\,k - r$, $d < r$. Hence in order that a construction shall be possible, k must be $< r$, point P must be so situated that $OP < r$ but $> 2\,k - r$. When $k > r$, the given ⊙ is tangent internally to the required ⊙, and therefore the given point P is outside the given ⊙.

193. For convenience we shall denote the parts of a triangle as follows: the sides by a, b, c; the angles opposite these sides by A, B, C; the altitude upon a by h_a (read h sub a), that upon b by h_b, that upon c by h_c; the medians by m_a, m_b, m_c; the bisectors of the angles by t_A, t_B, t_C.

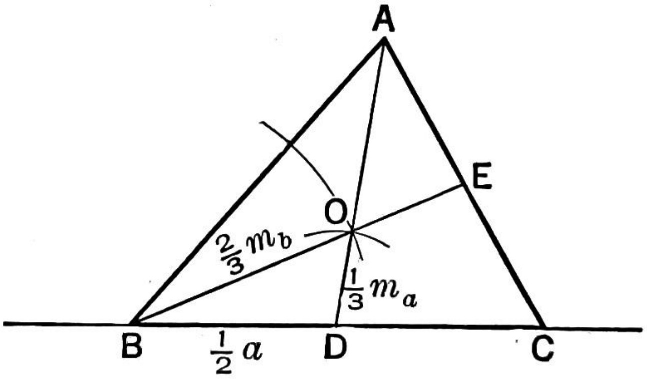
194. A triangle is determined when three independent parts are known. Note that three angles do not determine a triangle, since they constitute only two independent parts. Hence at least one of the three independent parts must be a line, the sum of lines or the difference of lines.

Ex. 272. Construct a triangle, having given a, m_a, m_b.



Analysis. Assume the problem done, and mark heavy the given parts. Since the medians of a triangle meet in a trisection point (124), *$B'O' = \frac{2}{3} m_b$, $O'D' = \frac{1}{3} m_a$. △ $B'O'D'$ is fully determined, since its three sides are known; and we make this △ the basis of the construction. Having constructed △ $B'O'D'$, we prolong $D'O'$ twice its length, and thus locate A'. By prolonging $B'D'$ its own length we locate C'. The required △ is now completely determined.*

Construction. Take BC equal to a, and find its midpoint D. Draw △ BOD with sides $\frac{1}{2} a$, $\frac{2}{3} m_b$, $\frac{1}{3} m_a$. Prolong DO twice its own length to A. Draw AB and AC. Then △ ABC is the required △.



Proof. △ ABC has BC equal to a, AD is the median to a and has the required length m_a (by construction). BO prolonged to meet AC at E must be the median to side b, since BE passes through O, the trisection point of median AD. Furthermore, BE has a length equal to m_b, since BO is $\frac{2}{3} m_b$ (124). ∴ △ ABC has the three given parts.

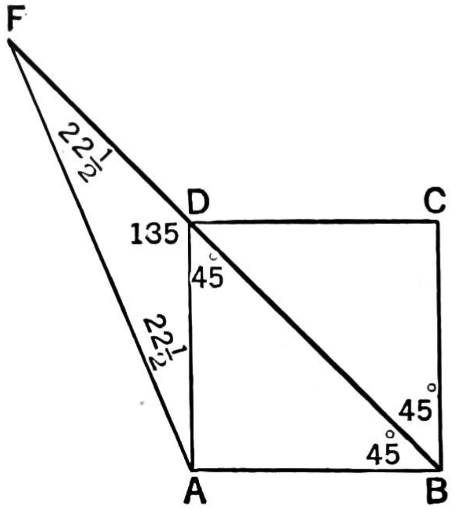
Q.E.D.

195. When a sum or difference is given, such sum or difference should be constructed in the analysis.

Ex. 273. Construct a square, having given the sum of the diagonal and one side.

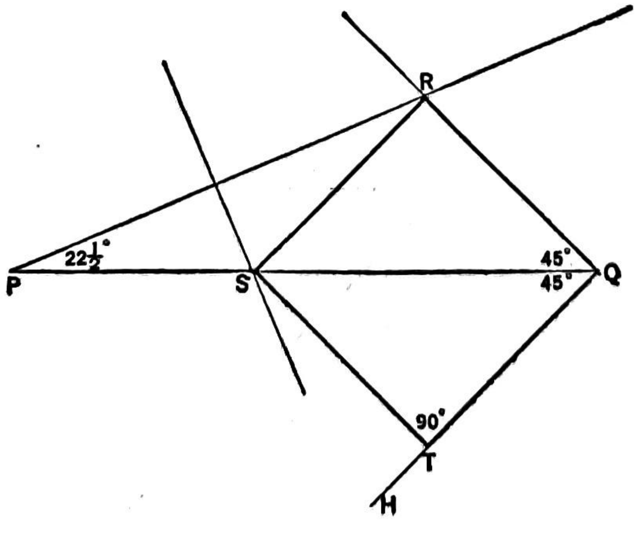
Given line PQ, which represents the length of the diagonal and one side of a square.

Required to construct the square.



Analysis. Assume the problem done. Prolong BD so that DF = DA. Then BF equals the sum of the diagonal and a side of the square. $\angle ABF = 45°$, $\angle AFB = 22\frac{1}{2}°$. $\therefore$ *△ AFB is determined, and thus AB, a side of the square, is found.*

Construction. On an indefinite line lay off PQ. At Q construct an $\angle$ of 45°. At P construct an $\angle$ of $22\frac{1}{2}°$. Let these lines intersect at R. Draw the $\perp$ bisector of PR, and let it cut PQ at S. Construct $\angle PQH$ equal to 45°. Draw $ST \perp HQ$. Then $QRST$ is the required square.

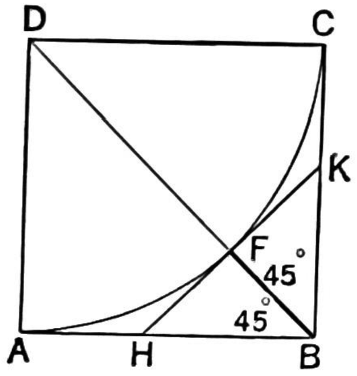


Proof. $SR = SP$ (77a). $\therefore \angle SRP = 22\frac{1}{2}°$. $\therefore \angle RSQ = 45°$. $\therefore \angle SRQ = 90°$, but $\angle TSQ = 45°$. $\therefore \angle RST = 90°$, and since $TS = TQ$ (73), the figure $QRST$ is a rectangle with two adjacent sides equal. In other words, it is a square, SQ is its diagonal, and $SQ + SR = SQ + SP = PQ$. $\therefore$ the sum of the diagonal and a side equals the given length PQ. Q.E.D.

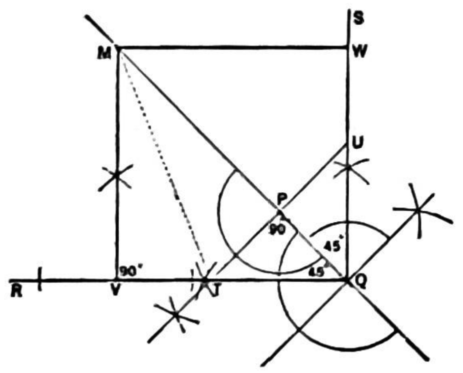
Ex. 274. Construct a square, having given the difference of the diagonal and one side.

Given line PQ, which represents the difference of the diagonal and one side of a square.

Required to construct the square.



Analysis. Assume the problem done, and let ABCD be the required square. An arc with D as center and DA as radius will cut BD in F and will be tangent to AB and BC. FB is the difference between the diagonal and a side of the square ABCD. A tangent to the arc at F is ⊥ FB and cuts AB in H and BC in K. △ BHK is determined, and we thus have part of the side AB and of the side BC. The remaining part HA equals the known distance HF and similarly KC equals KF (165). *Thus AB and BC, sides of the square, are found.*



Construction. On an indefinite line lay off PQ. At Q construct $\angle PQR = 45°$ and $\angle PQS = 45°$. Draw $TU \perp PQ$ at P, cutting QR in T and QS in U. On QR lay off TV equal to TP, and on QS lay off UW equal to PU. Draw $VM \perp QR$, and let it meet QP prolonged in M. Draw MW. Then $MVQW$ is the required square.

Proof. $\triangle TPQ \cong \triangle UPQ$. $\therefore TQ = UQ$, and $PT = PU$. $VQ = VT + TQ = TP + TQ = PU + UQ = UW + QU = QW$. $\angle VQM = 45°$. $\therefore \angle VMQ = 45°$. $\therefore VM = VQ$. $\therefore VM = QW$, but $VM \parallel QW$. $\therefore MVQW$ is a ▱ (89). $\therefore \angle M = 90°$, $\angle W = 90°$ and $MW = VQ$. $\therefore MVQW$ is a rectangle with all its sides equal. In other words, it is a square. MQ is its diagonal. $\triangle MVT \cong \triangle MPT$ (75). $\therefore MP = MV$. $PQ = MQ - MP = MQ - MV$. $\therefore$ the square $MVQW$ has the difference between its diagonal and a side equal to the given distance PQ.

Q.E.D.

196. We thus see that a logical method of attacking construction problems is to use analysis. The procedure may be summarized as follows:

(1) Assume the problem done, and draw freehand a clear figure which will resemble fairly well the ultimate drawing.

(2) Mark heavy all the parts that are given or that can be readily found from the given parts.

(3) Find some part of the figure (usually a triangle) that is fully determined, and make this the basis of the construction.

(4) Determine all other points of the figure.

(5) Make the actual construction, following the plan decided upon in (3) and (4).

(6) Prove that the construction satisfies all the given conditions.

(7) Discuss the construction as to possibility of solution, or as to number of solutions.

Problems of Construction

Ex. 275. Draw a line which shall be tangent to a given circle and parallel to a given line.

Ex. 276. Draw a line which shall be tangent to a given circle and perpendicular to a given line.

Ex. 277. Construct an isosceles triangle, having given the base and the vertex angle.

Ex. 278. Construct a right triangle, having given one leg and the altitude upon the hypotenuse.

Ex. 279. Construct a right triangle, having given the hypotenuse and the altitude upon the hypotenuse.

Ex. 280. Construct an isosceles triangle, having given one leg and the altitude on the leg.

Ex. 281. Construct an equilateral triangle, having given the radius of the circumscribed circle.

Sug. Use Ex. 252.

Ex. 282. Construct a parallelogram, having given one side and the two diagonals.

Ex. 283. Construct an isosceles triangle, having given the perimeter and the altitude to the base.

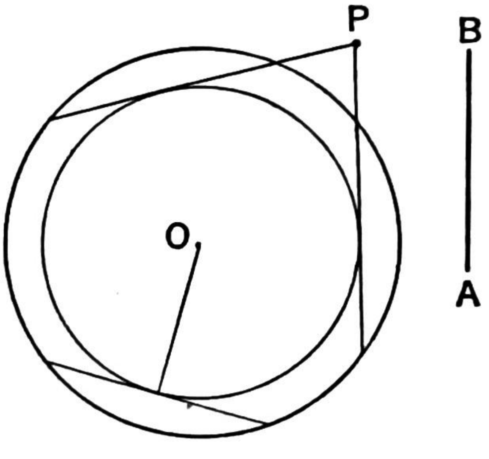
Ex. 284. Construct a triangle, having given b, c, m_c.

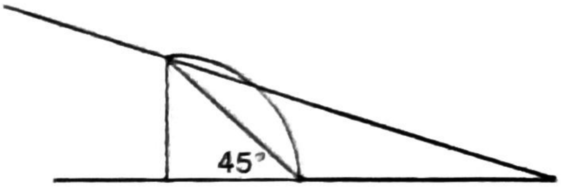
Ex. 285. In a triangle ABC draw DE parallel to the base BC, cutting the sides of the triangle in D and E, so that DE shall equal $DB + EC$.

Ex. 286. Construct a triangle, having given c, C, h_c.

Ex. 287. Through a given point inside or outside a given circle, draw a chord of given length.

Sug. From any point on the circle draw a chord whose length equals AB. Then use Ex. 249 and draw a tangent to the inner ⊙ from the point P.





Ex. 288. Construct a right triangle, having given an acute angle and the difference of the legs.

Ex. 289. Construct a triangle, having given A, B, t_c.

Ex. 290. Construct a triangle, having given A, c, m_c.

Ex. 291. Construct a triangle, having given c, m_c, h_b.

Ex. 292. Construct an isosceles triangle, having given the base and the radius of the inscribed circle.

Ex. 293. Construct a right triangle when the segments of the hypotenuse made by the bisector of the right angle are given.

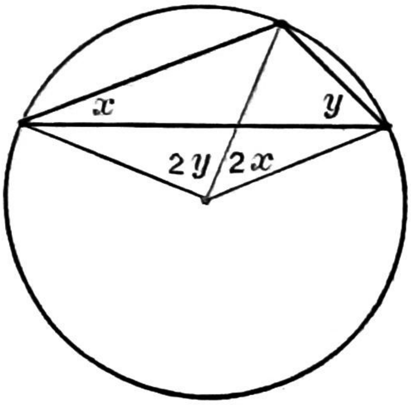
Ex. 294. In a given triangle construct a semicircle tangent to two sides of the triangle and having its diameter on the third side.

Ex. 295. Construct a tangent to a given circle, perpendicular to a given chord produced.

Ex. 296. Construct a line that shall make an angle of 30° with a given line and be tangent to a given circle.

Ex. 297. Given one angle of a rhombus and the diagonal drawn to the vertex of that angle; construct the rhombus.

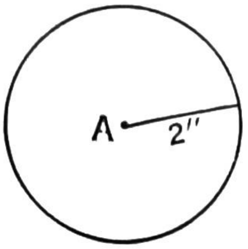
Ex. 298. In a given circle inscribe a triangle, having given two of its angles.



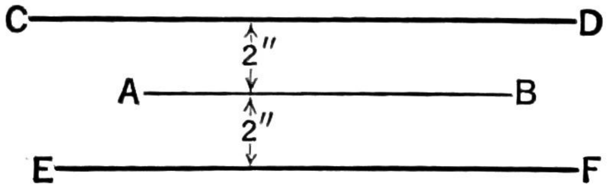
LOCI

197. Consider the following problems:

(1) Given a point A in a plane, required to locate all points in the plane that are $2''$ from A. Obviously the points will be situated in a circle whose center is A and whose radius is $2''$. That is, all points on the circle will be $2''$ from A, and these are the only points so situated.

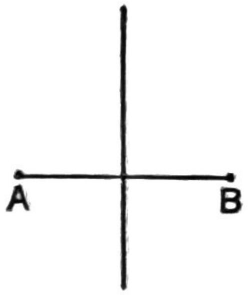


(2) Given an indefinite straight line AB in a plane, required to locate all points in the plane that are $2''$ from AB. Obviously the points will be situated in a straight line CD which is $\parallel AB$ and $2''$ from AB. That is, all points on CD will be $2''$ from AB. But these are not the only points so situated. In fact, all points in a line $EF \parallel AB$ and $2''$ from AB will also satisfy the requirement. Therefore the complete answer is: the points will be situated in two lines, each $\parallel AB$, one above it and one below it, and each $2''$ from it.

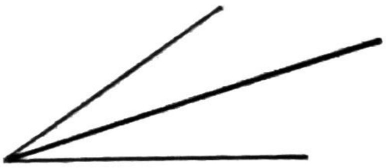


198. In the above examples, instead of saying "locate all points," we might have said "what is the *locus* of a point?" Locus is a Latin word that means place, position. Its plural is loci. Remembering that in Plane Geometry we confine all geometric figures to the plane, problem (1) above is usually stated as follows: What is the locus of a point two inches from a given point A? *Ans.* The locus is a *circle* whose center is A and whose radius is $2''$. Problem (2) above is: What is the locus of a point $2''$ from a given line AB? *Ans.* The locus is *two straight lines*, each $\parallel AB$, one above it and one below it, and each $2''$ from it.

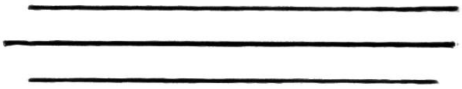
199. Other fundamental loci problems are:



(3) What is the locus of a point equidistant from two given points A and B? *Ans.* The locus is *the perpendicular bisector* of the line AB.



(4) What is the locus of a point within an angle equidistant from the sides of the angle? *Ans.* The locus is the *bisector* of the angle.

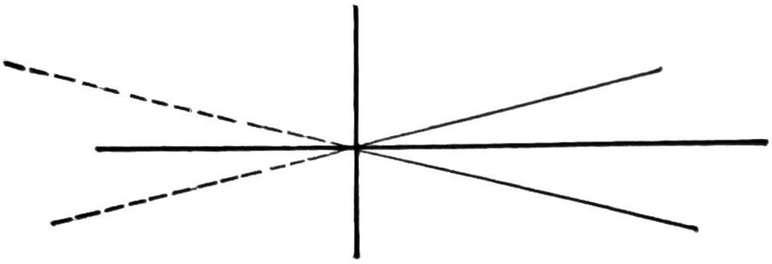


(5) What is the locus of a point equidistant from two given parallel lines? *Ans.* The locus is *the straight line* that is parallel to both and midway between them.

200. We thus see that a locus may be one or more lines, straight or curved. Furthermore, we must make sure that *all* points that satisfy the requirements have been accounted for. [See 197 (2).] This can be done by proving two things:

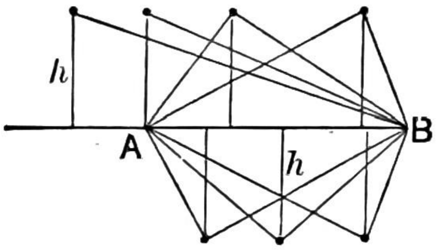
(*a*) Every point in the locus satisfies the given condition;
(*b*) Every point that satisfies the given condition is in the locus; or
(*a*) Every point in the locus satisfies the given condition;
(*c*) Every point not in the locus does not satisfy the given condition. Note that either of these combinations is necessary and sufficient to establish the proof of a locus. At times, one combination may be more readily proved than the other.

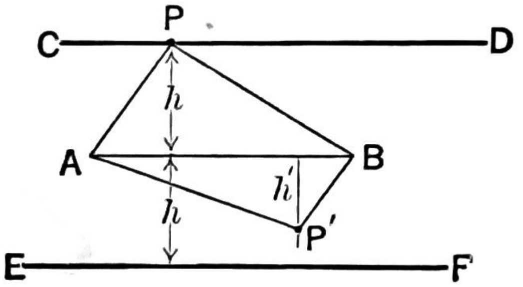
201. Prop. XIII of Book I may now be stated as follows: The locus of a point equidistant from the ends of a line is the perpendicular bisector of the line. Prop. XIV of Book I may be stated: The locus of a point within an angle equidistant from its sides is the bisector of the angle.



202. Note the phrase "within an angle." Had that been omitted, the locus would have been *two lines*, viz. the bisector of the given angle and the bisector of its supplementary adjacent angle.

203. In solving loci problems we must first determine the locus and then prove it. A good way to determine the locus is to locate three or more points that satisfy the condition. The situation of these points will usually suggest the locus.





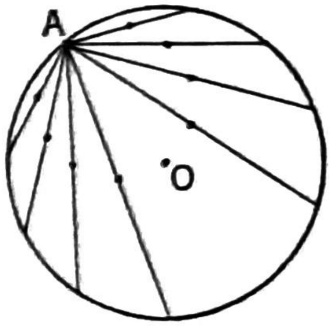
Ex. 299. What is the locus of the vertex of all triangles constructed on a given line AB as base and having a given altitude h? Experimentally we find, by constructing some triangles, that the vertices are probably situated in two lines parallel to the given line. Hence it would seem that the locus is two lines || the given line, one above it and one below it, at the distance h from it. But to make absolutely sure that this is the locus we must prove it. We shall use the (*a*) and (*c*) combination of 200.

Proof. 1. Let P be *any* point in CD or EF. Then △ APB satisfies the requirement, i.e. it has the given base AB and the given altitude h (|| lines are everywhere equidistant).

2. Let P' be *any* point not in CD or EF. Then △ ABP' does not satisfy the requirement, because it has the given

base but its altitude h' is less than h. Hence we have proved the two necessary and sufficient conditions to establish the locus.

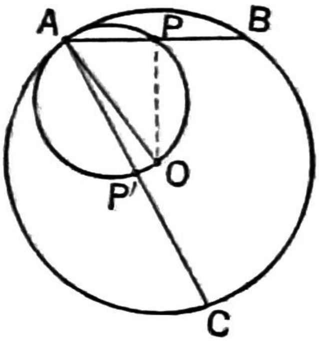
Ex. 300. What is the locus of the midpoints of all chords that can be drawn from a given point on a given circle?



Given ⊙ O with A a given point on the ⊙.

Find the locus of the midpoints of all chords that can be drawn from A.

Solution. By locating points, it would seem that the required locus is a circle constructed on AO as diameter.



Proof. 1. Let P be *any* point on the ⊙ whose diameter is AO. Draw AP, and prolong it till it meets the given ⊙ at B. Then P is the midpoint of chord AB. For, draw OP. Then $\angle APO$ is a right angle (177). $\therefore OP \perp AB$ and $\therefore P$ is the midpoint of AB (142).

2. Let P' be the midpoint of *any* chord AC. Draw OP'. Then $OP' \perp AC$. $\therefore \angle AP'O$ is inscribable in a semicircle whose diameter is AO. $\therefore P'$ lies in the supposed locus. Hence we have proved the two necessary and sufficient conditions to establish the fact that *the circle constructed with AO as diameter* is the required locus.

204. Usually, in the case of loci problems, an accurate drawing and statement of the locus is all that is required of the pupil. But where the question specifically calls for a proof, the proof must be given. Indeed the pupil should at all times be ready to prove geometric statements that are not axioms or fundamental assumptions.

Ex. 301. Find the locus of the midpoints of all chords of a given length that can be drawn in a given circle.

Ex. 302. Find the locus of the center of a circle which has a given radius and is tangent to a given line.

Ex. 303. Find the locus of the center of a circle which is tangent to a given line at a given point.

Ex. 304. Find the locus of the vertex of an isosceles triangle constructed on a given line as base.

Ex. 305. Find the locus of the center of a circle that passes through two given points.

Ex. 306. Find the locus of the vertex of a triangle that has a given base and a given angle opposite the base.

Sug. Use 190.

Ex. 307. Find the locus of a point at a given distance from a given circle. (The distance of a point from a circle is measured on a straight line that joins the point and the center of the circle.)

Ex. 308. Find the locus of the vertex of the right angle of a right triangle constructed on a given line as hypotenuse.

Ex. 309. Find the locus of the center of a circle that is tangent to two given lines. Consider the case when the lines are parallel; and also when the lines intersect.

Ex. 310. Find the locus of the center of a circle tangent to each of two given concentric circles.

Ex. 311. Find the locus of the center of a circle with a given radius and tangent to a given circle (*a*) internally; (*b*) externally.

Ex. 312. Find the locus of the center of a circle that is tangent to a given circle at a given point.

Ex. 313. What is the locus of the midpoint of a line drawn from one vertex of a given triangle to the opposite side?

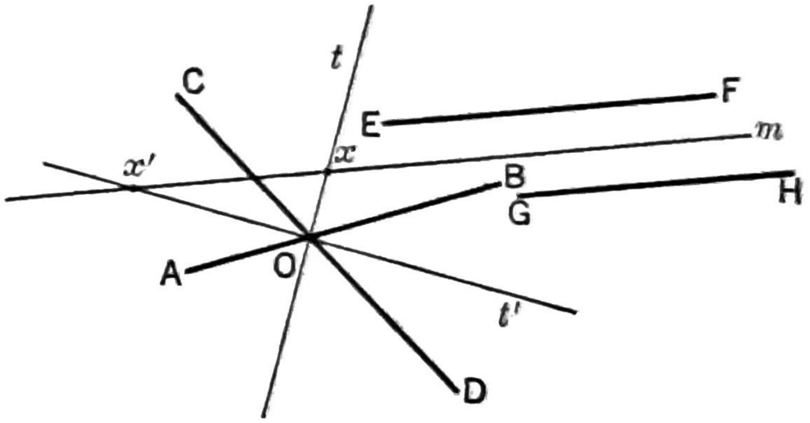
Ex. 314. What is the locus of the extremity of a tangent of given length to a given circle?

Ex. 315. What is the locus of the midpoint of a chord formed by a secant from a given external point to a given circle?

Intersection of Loci

205. When it is required to locate a point that satisfies more than one condition, we draw the locus for each condition and then note where the loci intersect. The points of intersection are the answer to the problem. The pupil should bear in mind that if the question asks to "locate a point," *all* points that fulfill the requirements must be found.

Ex. 316. Find a point equidistant from two given intersecting lines and also equidistant from two given parallel lines.



Given the intersecting lines AB and CD, also $EF \parallel GH$.

Required to locate a point that is equidistant from AB and CD and at the same time equidistant from EF and GH.

Solution. To be equidistant from AB and CD, the point must be somewhere in line t or t', the bisectors of $\angle BOC$ and $\angle BOD$ (202). To be equidistant from EF and GH, the point must lie somewhere in line m, midway between EF and GH and $\parallel$ both [199 (5)]. $\therefore$ the required point must be where t and t' are cut by m, viz. x and x'.

Discussion. If $EF \parallel t$ or t', then only one point can be found. If EF and GH are so situated that m passes through O, then O is the required point. If EF and GH are so situated that m coincides with t or t', then there is a whole line full of points that satisfy the requirements. $\therefore$ the possibilities to this problem are: one point, two points, an infinite number of points lying in t or t'.

Note: Lines t, t', and m should be constructed with ruler and compasses.

Ex. 317. Find a point which lies in one side of a given triangle and is equidistant from the other two sides.

Ex. 318. Find a point equidistant from two given points and also equidistant from two given parallel lines.

Ex. 319. Find a point equidistant from three given points.

Ex. 320. Find a point equidistant from two given points and at a given distance from a third given point.

Ex. 321. In a given circle, find a point at a given distance k from a given point A.

Ex. 322. Find a point equidistant from two given points and also equidistant from two given intersecting lines.

Ex. 323. Find a point equidistant from two given parallel lines and at the given distance k from a given point P.

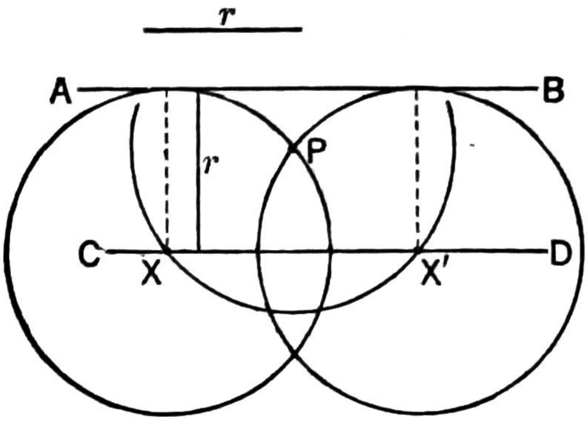
Ex. 324. Find a point at the given distance d from a given line AB, and equidistant from two given parallel lines CD and EF.

Ex. 325. Given an angle, a point K on one of its sides, and a line segment d. Find a point equally distant from the sides of the angle, and at the distance d from K.

Ex. 326. Find all points that are equidistant from two given concentric circles and also equidistant from two given points on the larger circle.

206. Certain construction problems can be readily solved by the intersection of loci.

Ex. 327. Construct a circle having a given radius, passing through a given point and tangent to a given line.



Given line AB, point P, and length r.

Required to draw a circle of radius r which shall be tangent to AB and pass through P.

Solution. Since the required circle is to pass through P and have a radius r, its center must lie somewhere on a circle whose center is P and whose radius is r. Furthermore, since the required circle is to be tangent to the given line AB, its center must lie somewhere in a line $CD \parallel AB$ at the distance r from AB. The points of intersection of these two loci, X and X', are the centers of two circles that satisfy the requirements. Hence with X as center and r as radius draw a ⊙; also with X' as center and r as radius draw a circle, and the problem is done.

Proof. $XP = r$, $X'P = r$. ∴ the Ⓢ pass through P and have the radius r. X and X' are at the distance r from AB ($\parallel$s are everywhere equidistant). ∴ the Ⓢ will pass through the feet of the perpendiculars dropped from X and X' to AB. ∴ AB is tangent to the Ⓢ (159).

Discussion. If we denote the distance from P to AB by d, then

if $r > d$ there are two solutions
if $r = d$ there are two solutions.
if $r < d$ but $> \frac{1}{2}d$ there are two solutions.
if $r = \frac{1}{2}d$ there is one solution.
if $r < \frac{1}{2}d$ there is no solution.

Ex. 328. Draw a circle which shall pass through a given point and be tangent to a given line at a given point.

Ex. 329. Draw a circle having a given radius and passing through two given points.

Ex. 330. Draw a circle tangent to two given parallel lines and passing through a given point unequally distant from the two lines.

Ex. 331. Draw a circle tangent to two given intersecting lines and touching one of them at a given point.

Ex. 332. Draw a circle passing through a given point and tangent to a given circle at a given point.

Ex. 333. Given a circle K and a line L outside the circle. Draw a circle tangent to K so that its center shall lie in L and its radius shall be R.

Ex. 334. Construct a triangle, having given c, h_c, m_c.

Ex. 335. Construct a triangle, having given a, m_a, A.

Miscellaneous Exercises

Ex. 336. Prove that the circle described on one of the sides of an equilateral triangle as diameter bisects the other sides of the triangle.

Ex. 337. Prove that if two tangents drawn from an external point to a circle form an angle of 120°, the distance from the point to the center of the circle is equal to the sum of the tangents.

Ex. 338. The radius of the circle inscribed in an equilateral triangle is one third of the altitude of the triangle.

Ex. 339. The angle formed by two tangents to a circle is twice the angle formed by the radius to the point of contact and the chord of contact.

Ex. 340. AOD is the diameter of a circle whose center is O. B is any point on the circle. At B a tangent is drawn, and OP is drawn parallel to AB meeting the tangent at P. Prove that PD is tangent to the circle.

Ex. 341. Lines from the center of a circle to the ends of a leg of the circumscribed trapezoid form a right angle.

Ex. 342. Prove that if the inscribed and circumscribed circles of a triangle are concentric, the triangle is equilateral.

Ex. 343. The sum of the lines drawn from the extremities of any diameter of a circle perpendicular to a tangent is equal to a diameter of the circle.

Sug. Apply Ex. 160.

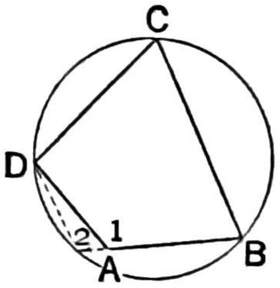
Ex. 344. Two lines are tangent to a circle from an external point. Draw a diameter from one point of tangency and join its other extremity with the other point of tangency. Prove that this line is parallel to the line joining the external point with the center of the circle.

Ex. 345. Prove that tangents to a circle at the extremities of any two diameters form a rhombus.

Ex. 346. *AB* and *CD* are unequal parallel chords. Prove that *AD* and *BC* will intersect on the diameter perpendicular to the chords, or on the diameter produced.

Ex. 347. A circle may be circumscribed about any quadrilateral whose opposite angles are supplementary.

Outline of proof. A circle may always be drawn through three consecutive vertices such as *B*, *C*, *D*. This ⊙ must either pass below *A*, above *A*, or through *A*. If it could pass below *A*, ∠ 1 would have to equal ∠ 2 (since each is the supplement of ∠ *C*). This is absurd (see 49). In like manner it can be shown that the ⊙ cannot pass above *A*. ∴ it must pass through *A*.

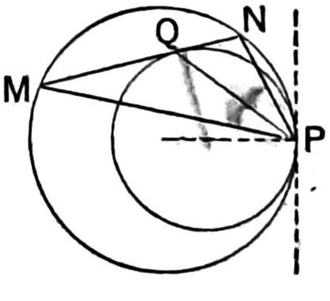


Ex. 348. Prove that the right angle of any right triangle is bisected by a line drawn from its vertex to the middle of the square constructed on the hypotenuse.

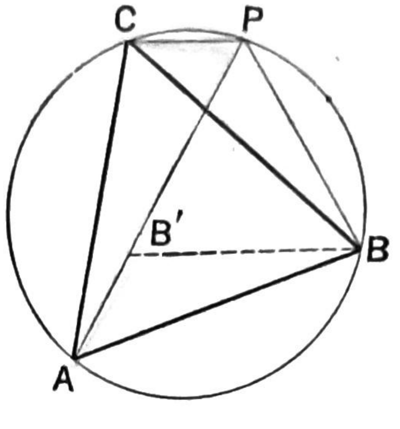
Sug. Use Ex. 347.

Ex. 349. If one circle touches another internally at *P*, and a tangent to the first at *Q* intersects the second in *M* and *N*, then ∠ *MPQ* = ∠ *NPQ*.

Sug. Draw the common tangent, and study measure of angles.

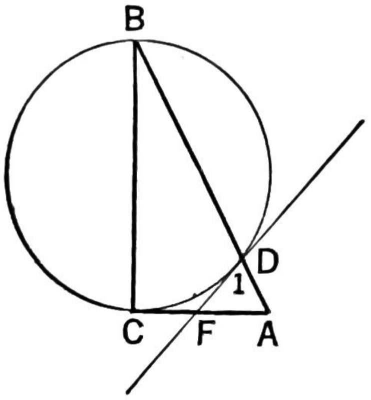


Ex. 350. If a circle is circumscribed about an equilateral triangle and from any point in the circumference straight lines are drawn to the three vertices, one of these lines is equal to the sum of the other two.



Sug. Prove $PA > PB$, and mark off $PB' = PB$. Draw BB'. Then prove $\triangle PBC \cong \triangle ABB'$ (by s. a. s.). $\therefore PC = AB'$. $\therefore PA = PB + PC$.

Ex. 351. If one leg of a right triangle is the diameter of a circle, the tangent at the point where the circle cuts the hypotenuse bisects the other leg.



Sug. $\angle A = \angle 1$ (each is measured by $\frac{1}{2}\widehat{BD}$). $\therefore FA = FD$. But $FC = FD$. $\therefore FA = FC$.

Ex. 352. Two fixed circles touch each other externally, and a circle of variable radius touches both externally. Prove that the difference of the distances from the center of the variable circle to the centers of the fixed circles is constant.

Sug. Use last part of 169b, and prove that the constant referred to is equal to the difference of the radii of the fixed circles.

Ex. 353. The line joining the midpoints of the non-parallel sides of a circumscribed trapezoid is equal to one fourth of the perimeter of the trapezoid.

Sug. Apply Ex. 160 and Ex. 226.

Ex. 354. *ABCD* is a quadrilateral inscribed in a circle. *AD* and *BC* are extended to meet at *P*. A circle is circumscribed about △ *ABP*, and a tangent is drawn to this circle at *P*. Prove that the tangent is parallel to *CD*.

Ex. 355. If *AB* and *AC* are tangents to a circle from a point *A*, touching the circle at *B* and *C* respectively, and if *D* is any point on the minor arc *BC*, then ∠ *BDC* = 90° + $\frac{1}{2}$ ∠ *A*.

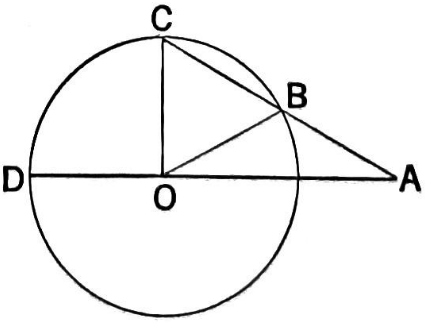
Ex. 356. If *AB* and *AC* are tangents to a circle from a point *A*, touching the circle at *B* and *C* respectively, and if *D* is any point on the major arc *BC*, then ∠ *BDC* = 90° − $\frac{1}{2}$ ∠ *A*.

Ex. 357. Two radii at right angles to each other are extended to meet a tangent, and from these points of intersection tangents are drawn to the circle. Prove that these two tangents are parallel.

Ex. 358. Two tangents are drawn to a circle *O* at the ends of a diameter, and intercept a segment *PQ* on a third tangent. Prove that ∠ *POQ* is a right angle.

Ex. 359. *O* is the center of a circle, *OA* any radius, and *B* any point on a radius perpendicular to *OA*. Line *AB* produced meets the circle at *C* and the tangent at *C* meets *OB* produced at *D*. Prove that △ *BCD* is isosceles.

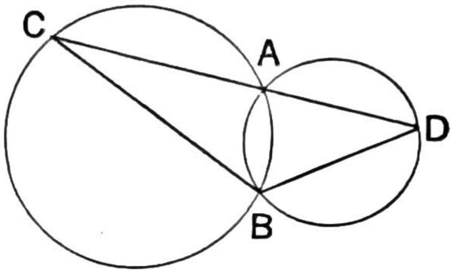
Ex. 360. *AOD* and *ABC* are straight lines. *O* is the center of the circle and *OB* = *AB*. Prove that ∠ *A* equals $\frac{1}{3}$ ∠ *COD*.



Ex. 361. Circles constructed on two sides of a triangle as diameters will intersect on the third side or on the third side produced.

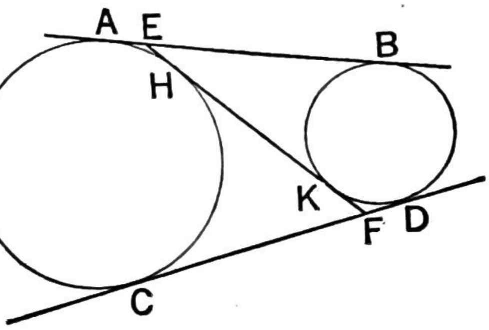
Sug. Assume that they do not, and then show that this results in having two lines from an external point perpendicular to the same line.

Ex. 362. Two circles intersect at the points A and B. Through A a variable secant is drawn cutting the circles in C and D. Prove that the $\angle CBD$ is constant for all positions of the secant.



Sug. Prove $\angle C + \angle D$ is constant.

Ex. 363. The two common external tangents to two circles intercept on a common internal tangent a segment EF equal to the external tangent AB.



Sug. Starting with $AB = CD$, prove $EH = KF$.

Ex. 364. AB is a fixed chord of a circle and CD is any diameter. Prove that the sum of the perpendiculars from C and D to AB is constant.

Ex. 365. Two circles intersect at A, B. Through C, any point on one circle, lines CAD, CBE are drawn cutting the other circle in D, E. Prove that the chord DE is parallel to the tangent at C.

Ex. 366. The radii of two concentric circles are a and b, respectively. Find the radius of a third circle which shall be tangent to both the concentric circles and contain the smaller.

Ex. 367. If a circle is inscribed in a triangle whose sides are 8″, 13″, 17″ respectively, find the length of each of the segments into which the sides are divided by the points of tangency.

Ex. 368. Find a point equidistant from two adjacent sides of a quadrilateral, and also equidistant from the other two sides.

Ex. 369. From a given point on a circle draw a chord that is bisected by a given chord.

Ex. 370. In a given circle draw a chord equal to a given line and parallel to another given line.

Sug. This problem reduces to Ex. 275.

Ex. 371. In a diameter of a circle, produced indefinitely, find a point such that the tangent drawn from it to the given circle shall be of given length.

Ex. 372. In a diameter of a circle, produced indefinitely, find a point such that two tangents drawn from it to the circle shall form a given angle.

Ex. 373. Find a point such that the sum of the tangents drawn from it to a given circle shall equal the sum of two given lines.

Sug. Apply Ex. 371.

Ex. 374. Divide a given circle into two segments such that one of them shall contain an angle twice as great as the other.

Ex. 375. Through one of the points of intersection of two unequal circles draw a line terminated by the circles and bisected by the point of intersection.

Sug. From the midpoint of the centers-segment draw a line to the point of intersection of the circles, etc. DRAW THE ⊥S from the centers to the 2 = chords. PROVE: the line to the point of intersection is ∥ to them.

Ex. 376. Draw a line meeting the sides of ∠ A in B, C so that BC equals a given length k and $AB = AC$.

Ex. 377. Draw a diameter of a given circle at a distance k from a given point on the circle.

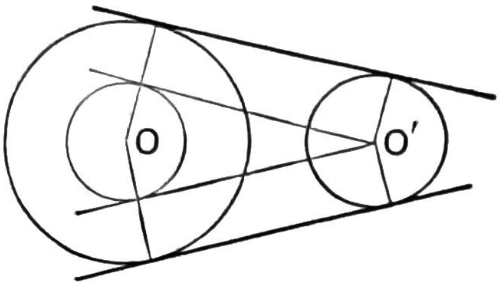
Ex. 378. Draw two circles of given radii which shall be tangent to each other externally and also be tangent to a given line on the same side of the line.

Ex. 379. Draw a circle of given radius tangent to two given circles.

Ex. 380. Draw a circle tangent to a given line and to a given circle at a given point.

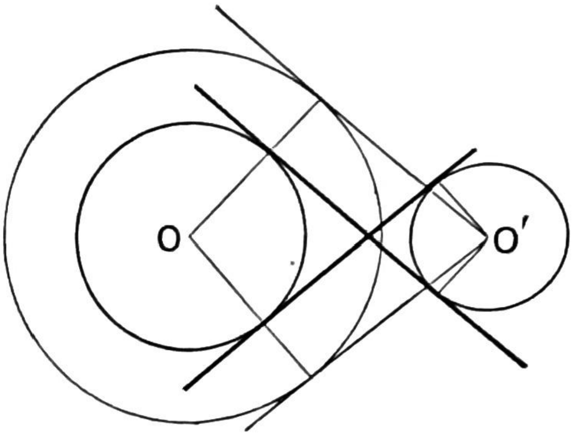
Ex. 381. In a square $ABCD$ construct an equilateral triangle AEF so that E, F shall lie on sides of the square.

Ex. 382. Construct a common external tangent to two unequal circles.



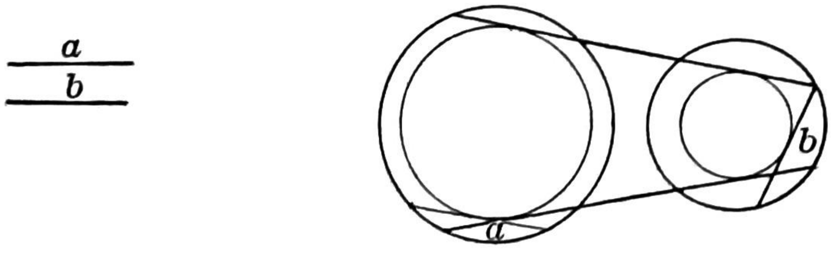
Sug. Draw a circle with radius equal to the difference of the radii of the given circles. Draw tangents from O' to this circle, etc.

Ex. 383. Construct a common internal tangent to two unequal circles.



Sug. Draw a circle with radius equal to the sum of the radii of the given circles. Draw tangents from O' to this circle, etc.

Ex. 384. Draw a secant to two given circles so that the chords shall have given lengths a, b.



Sug. This problem reduces to Ex. 382.

Ex. 385. Construct a triangle, given a, b, h_c.

Ex. 386. Construct a triangle, given A, B, h_a.

Ex. 387. Construct a triangle, given A, B, t_A.

Ex. 388. Construct a triangle, given C, t_C, h_c.

Ex. 389. Construct a triangle, given a, b, m_c.

Ex. 390. Construct a rectangle having given a diagonal and the sum of two adjacent sides. Sug. Apply 191.

Ex. 391. Construct a rhombus having given an angle, and the sum of a side and the longer diagonal. Sug. Apply 104.

Ex. 392. Construct an isosceles trapezoid having given a diagonal, a leg, and the altitude.

Ex. 393. Construct an isosceles trapezoid having given the bases and the altitude.

Sug. In trapezoid constructions it is often helpful in making the analysis drawing to draw a line through a vertex ∥ the other leg or ∥ one of the diagonals.

Ex. 394. Construct an isosceles trapezoid having given the bases and one diagonal.

Ex. 395. Construct a trapezoid having given the four sides.

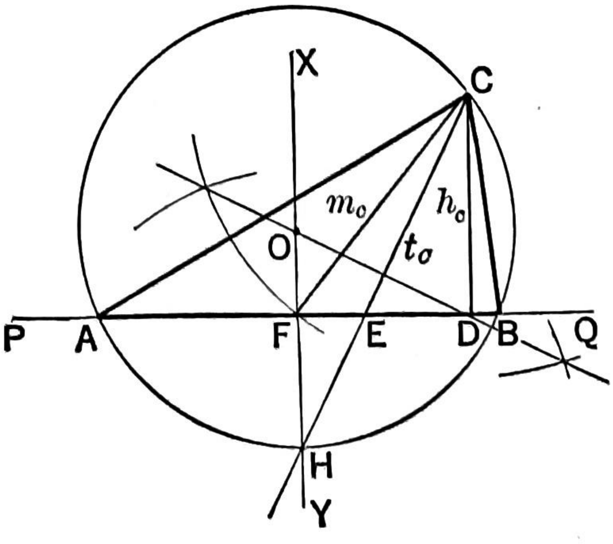
Ex. 396. Construct a trapezoid having given the greater base, the diagonals, and the angle between the diagonals.

Ex. 397. Construct a trapezoid having given the bases and the diagonals.

Ex. 398. Construct a triangle having given m_c, h_c, t_c.

Sug. Find the center of the circumscribed circle.

Solution. At any point D of an indefinite line PQ draw $DC \perp PQ$ and of length h_c. With C as center and t_c as radius locate E. Similarly locate F. At F draw XY perpendicular to PQ and let it meet CE produced at H. Where the $\perp$ bisector of CH meets XY is the center of the required circle. With O as center and OH as radius draw a circle. This circle cuts PQ in A and B. Then ABC is the required triangle.



Ex. 399. A series of parallelograms stand on the same base between the same parallels. Find the locus of the intersection of the diagonals.

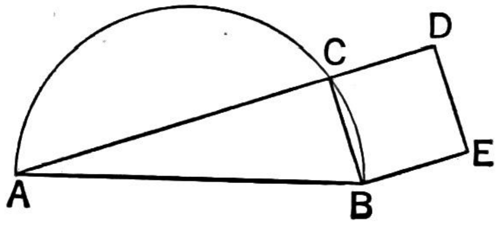
Ex. 400. A given line revolves on a given point A as a pivot. Find the locus of the foot of the perpendicular dropped on this line from a second point B.

Ex. 401. On a given line as a base, a system of rhombuses is constructed. Find the locus of the intersection of their diagonals.

Ex. 402. Find the locus of points of contact of tangents drawn from a fixed point to a system of concentric circles.

Ex. 403. A series of circles are tangent to each other at the same point. Find the locus of points of contact of tangents drawn to these circles from any point in the common tangent.

Ex. 404. ABC is a right triangle with the right angle at C. $BCDE$ is the square constructed on side BC. Find the locus of the vertex D as C moves about the semicircle $BC'A$.



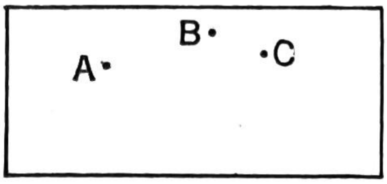
Sug. $\angle ADB$ is 45°. ∴ this example reduces to Ex. 306.

Ex. 405. The legs of a right triangle are given in position, its hypotenuse in length. Find the locus of the midpoint of the hypotenuse. Sug. Apply Ex. 100.

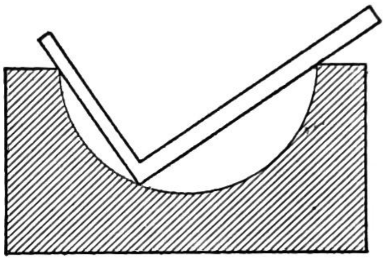
Ex. 406. AB is any diameter of a circle, and C, D are two fixed points on the circle. Find the locus of the intersection of AC and BD.

Sug. Prove that the angle formed by AC and BD is constant, and then apply Ex. 306.

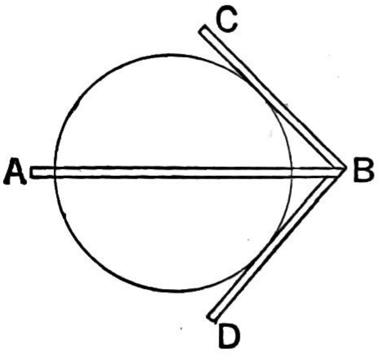
Ex. 407. AB and CD are non-intersecting chords of the same circle, and each is equal to the side of the inscribed square. AB is fixed, but CD is movable. AD and BC intersect in P. Find the locus of P.



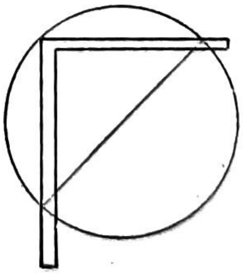
Ex. 408. *A*, *B*, *C* represent three towns on a map. Locate a school that will be equally distant from the three towns. If the scale of the map is 1″ = 5 miles, how far will the school be from each of the towns?



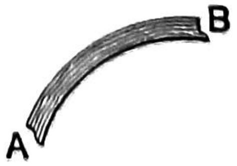
Ex. 409. Pattern-makers use a carpenter's square to test the accuracy of a semicircular groove. If the vertex touches the curve at all points while the sides rest on the two edges, it is a semicircle. Why?



Ex. 410. The center of a circular disk may be found by use of the instrument shown in the adjoining figure. It consists of three rods so joined that *AB* bisects the angle formed by *BC* and *BD*. If the instrument is placed as here shown, why will *AB* pass through the center of the circle? Hence how may the center be located?



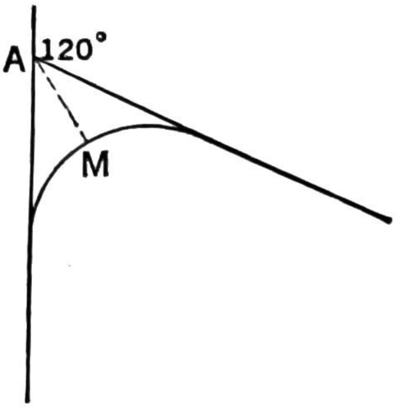
Ex. 411. Show how a carpenter's square may be used to find the center of a circle. What geometric principle is here used?



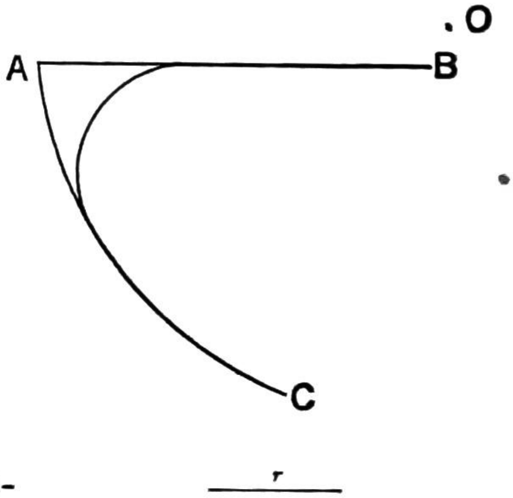
Ex. 412. *AB* represents a fragment of a wheel. From this fragment, how may the diameter of the wheel be found?

Ex. 413. Three circular disks are placed so that each is tangent to the other two. If the distances between their centers are 0.7″, 0.8″, 0.9″ respectively, find the diameter of each disk

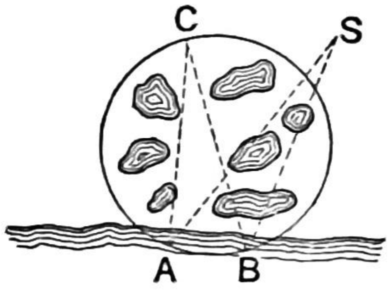
Ex. 414. Two straight tracks meet at an angle of 120°. It is required to round off the corner with a circular arc whose radius is 125 ft. Make an accurate drawing; also calculate the distance AM (M is the midpoint of the curve).



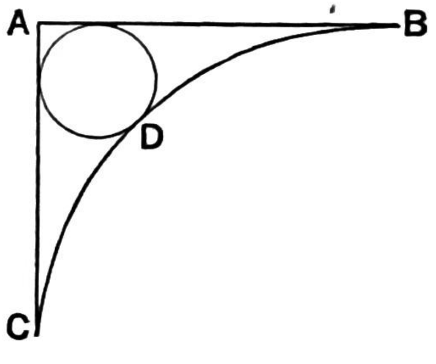
Ex. 415. AB is a straight street and AC is a circular street (center O). It is required to round off the corner with a circular arc whose radius shall equal a given line r. Make an accurate drawing.



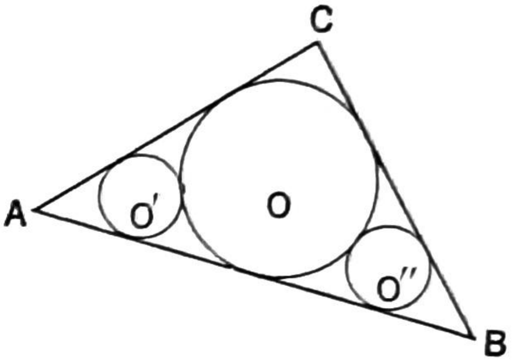
Ex. 416. A and B are two lighthouses on shore, and it is known that dangerous rocks are situated within a circle that passes through A and B. $\angle ACB$ is called the "horizontal danger angle." Show that if a ship S moves so that $\angle S$ is always less than $\angle C$, it is sure of avoiding the rocks.



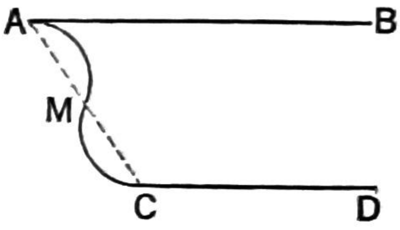
Ex. 417. The adjoining figure shows a rafter design. $AB \perp AC$. $\overset{\frown}{CDB}$ is tangent to the sides of A. Construct the small circle so that it shall be tangent to AB, AC, and $\overset{\frown}{CDB}$.



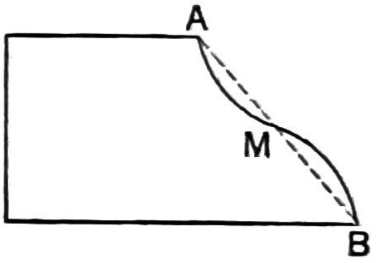
Ex. 418. The adjoining figure shows a window and rafter design. ABC is a right isosceles $\triangle$. Inscribe $\odot O$ in $\triangle ABC$ and make Ⓢ O' and O'' tangent to $\odot O$ and to two sides of the $\triangle$.



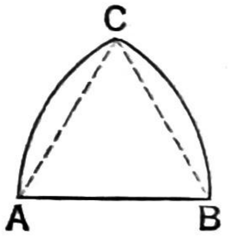
Ex. 419. Draw the molding here shown. $\widehat{AM}$ is tangent to AB at A, $\widehat{CM}$ is tangent to CD at C, M is the midpoint of AC.



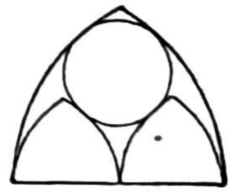
Ex. 420. Draw the molding here shown. M is the midpoint of AB. Each arc has a radius equal to AM.



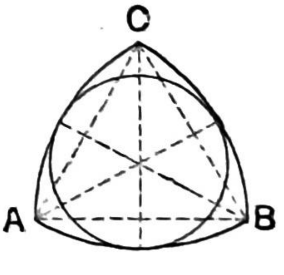
Ex. 421. ABC is an equilateral $\triangle$. B is the center of $\widehat{AC}$, A is the center of $\widehat{BC}$. The figure composed of AB, $\widehat{AC}$, and $\widehat{BC}$ is called an equilateral Gothic arch. AB is its span. The Gothic arch plays an important part in the architecture of cathedrals.



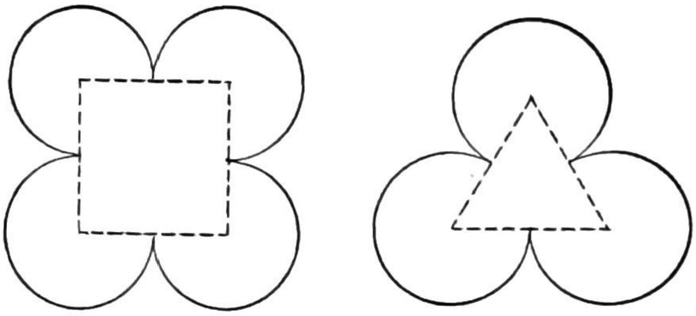
Construct three equilateral Gothic arches, each tangent to the other two. Then draw a circle tangent to the three arches.



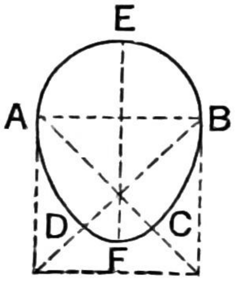
Ex. 422. Construct the window design here shown. $\triangle ABC$ is equilateral. Arcs BC, AC, AB are drawn with A, B, C as centers respectively and radius equal to AB. The circle is tangent to arcs AB, BC, AC.



Ex. 423. Draw the trefoil and quatrefoil as here shown.



Ex. 424. From an engineering viewpoint an egg-shaped sewer is preferable to a circular one. The figure here shown represents the cross-section of an egg-shaped sewer. It is made up of four circular arcs. Note that $AB = AC = BD$. Make an accurate drawing of this figure.



BOOK III

PROPORTIONAL LINES AND SIMILAR POLYGONS

PROPORTION

207. To prove the propositions of Book III it will be necessary to make use of some fundamental properties of proportion. These properties are purely algebraic relations, and belong properly to the study of algebra. But since we shall need them as tools for our work in geometry, a brief outline of proportion is here presented.

208. The *ratio* of two numbers is the relation expressed by dividing one number by the other. Thus the ratio of 5 to 10 is $\frac{5}{10}$ or $\frac{1}{2}$. It is written $5:10$ or $\frac{5}{10}$. In other words, a ratio is a fraction. The numerator is called the *antecedent*, the denominator is called the *consequent.*

209. A *proportion* is an equality of two ratios. Thus $\frac{5}{10} = \frac{3}{6}$ is a proportion. It asserts that the ratio of $5:10$ equals the ratio of $3:6$. A proportion is merely an equation. The above proportion is also written $5:10 = 3:6$ (read 5 is to 10 as 3 is to 6).

210. Note that a proportion involves four numbers. These numbers are called *terms* of the proportion. The first and fourth terms are called *the extremes,* the second and third terms are called *the means.* Thus in the above proportion 5 and 6 are the extremes, 10 and 3 are the means.

211. The fourth term of a proportion is called the *fourth proportional* to the first three terms. Thus 6 is the fourth proportional to 5, 10, and 3.

212. If a proportion has the two means equal, either mean is called the *mean proportional* between the extremes. Thus, in the proportion $4:6 = 6:9$, 6 is called the mean proportional between 4 and 9.

213. Using general numbers, if a, b, c, d are four numbers in proportion, then $\frac{a}{b} = \frac{c}{d}$; multiplying both sides by bd, we get $a \times d = b \times c$. Hence *in any proportion the product of the extremes equals the product of the means.*

214. Conversely if $a \times d = b \times c$; dividing both sides by bd, we get $\frac{a}{b} = \frac{c}{d}$, or dividing both sides of $b \times c = a \times d$ by ac, we get $\frac{b}{a} = \frac{d}{c}$. Hence *if the product of two quantities equals the product of two others, either pair may be made the means and the other the extremes of a proportion.*

215. If $\frac{a}{b} = \frac{c}{d}$, multiplying both sides by $\frac{b}{c}$, we get $\frac{a}{c} = \frac{b}{d}$. Hence *if four quantities are in proportion they are in proportion by alternation,* i.e. the first is to the third as the second is to the fourth.

216. If $\frac{a}{b} = \frac{c}{d}$, then $bc = ad$; dividing both sides by ac, we get $\frac{b}{a} = \frac{d}{c}$. Hence *if four quantities are in proportion they are in proportion by inversion,* i.e. the second is to the first as the fourth is to the third.

217. If $\frac{a}{b} = \frac{c}{d}$,

then $\frac{a}{b} + 1 = \frac{c}{d} + 1$,

or $\frac{a+b}{b} = \frac{c+d}{d}$.

Also, since $\frac{b}{a} = \frac{d}{c}$,

then $1 + \frac{b}{a} = 1 + \frac{d}{c}$,

or $\frac{a+b}{a} = \frac{c+d}{c}$.

Hence *if four quantities are in proportion they are in proportion by addition,* i.e. the sum of the first two is to the first or second as the sum of the last two is to the third or fourth.

218. If $\frac{a}{b} = \frac{c}{d}$,

then $\frac{a}{b} - 1 = \frac{c}{d} - 1$,

or $\frac{a-b}{b} = \frac{c-d}{d}$.

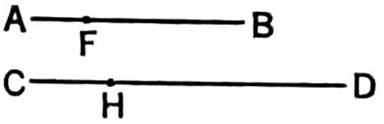
Also, since $\frac{b}{a} = \frac{d}{c}$,

then $1 - \frac{b}{a} = 1 - \frac{d}{c}$,

or $\frac{a-b}{a} = \frac{c-d}{c}$.

Hence *if four quantities are in proportion they are in proportion by subtraction,* i.e. the difference of the first two is to the first or second as the difference of the last two is to the third or fourth.

219. If $\frac{a}{b} = \frac{c}{d} = \frac{e}{f} = \frac{g}{h} = r$, then $a = br$, $c = dr$, $e = fr$, $g = hr$. $\therefore a + c + e + g = r\,(b + d + f + h)$, or $\frac{a + c + e + g}{b + d + f + h} = r = \frac{a}{b}$. Hence *in a series of equal ratios the sum of the antecedents is to the sum of the consequents as any antecedent is to its consequent.*



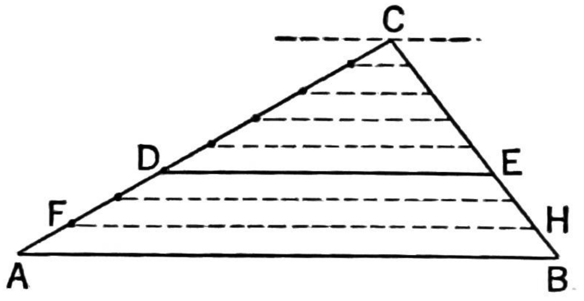
220. In the annexed drawing, if $AF = \frac{1}{4}AB$ and $CH = \frac{1}{4}CD$, then $AF : FB = CH : HD$, because each ratio equals $\frac{1}{3}$. We say that AB and CD are divided *proportionally.* In general, any two lines are divided proportionally if the ratio of the segments of one line equals the ratio of the segments of the other line.

221. The terms of the above proportion are really *numbers* which represent the numerical measures of the lines. This is true of any proportion involving lines.

PROPORTIONAL SEGMENTS

Proposition I. Theorem

222. A line drawn through two sides of a triangle parallel to the third side divides the two sides proportionally.



Given $\triangle ABC$ with $DE \parallel AB$, cutting AC in D and BC in E.

Prove $AD : DC = BE : EC$.

Plan is to prove each ratio equal to a third ratio.

Proof. Let AD and DC have a common measure AF, and let $AD = 3\,AF$ and let $DC = 5\,AF$. Then

(1) $\dfrac{AD}{DC} = \dfrac{3}{5}$	(1) Def. of ratio.
(2) Through the points of division on AC draw lines $\parallel AB$.	(2) 100.
(3) Then these lines divide BE into 3 parts and EC into 5 parts, each equal to BH.	(3) 108.
(4) $\therefore \dfrac{BE}{EC} = \dfrac{3}{5}$	(4) Def. of ratio.
(5) $\therefore \dfrac{AD}{DC} = \dfrac{BE}{EC}$	(5) Ax. 1.

Q.E.D.

223. The proof of the above theorem is the same if any other numbers are used. When AD and DC have no common measure the theorem is still true, but its proof requires a knowledge of the theory of limits. The pupil will therefore assume it to be true.

224. *Cor.* 1. One side of a triangle is to either of its segments cut off by a line parallel to the base as the other side is to the corresponding segment.

Proof. Since $AD : DC = BE : EC$ (222)
$AD + DC : AD = BE + EC : BE$ (217)
$\therefore AC : AD = BC : BE$
also $AD + DC : DC = BE + EC : EC$ (217)
$\therefore AC : DC = BC : EC$

225. *Cor.* 2. Three or more parallel lines cut off proportional segments on any two transversals.

Proof. $\frac{a}{b} = \frac{d'}{e'}$ ①

$\therefore \frac{a}{b} = \frac{d}{e}$ ②

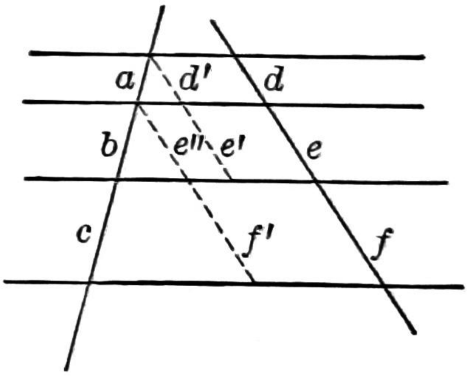
by alternation, $\therefore \frac{a}{d} = \frac{b}{e}$ ③

also $\frac{b}{c} = \frac{e''}{f'}$ ④

$\therefore \frac{b}{c} = \frac{e}{f}$ ⑤

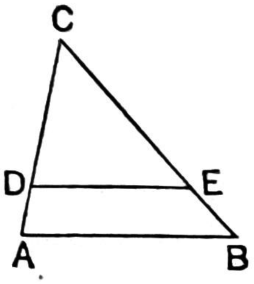
by alternation, $\therefore \frac{b}{e} = \frac{c}{f}$ ⑥

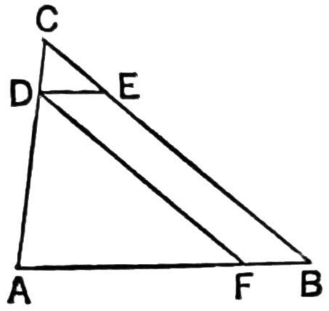
from ③ and ⑥, $\therefore \frac{a}{d} = \frac{b}{e} = \frac{c}{f}$



Exercises

Ex. 425. In the annexed drawing $DE \parallel AB$ $AC = 8''$, $BC = 10''$, $AD = 2''$. Find the length of BE and of EC. If the altitude on AB were drawn, what would be the ratio of the segments of the altitude formed by the line DE?

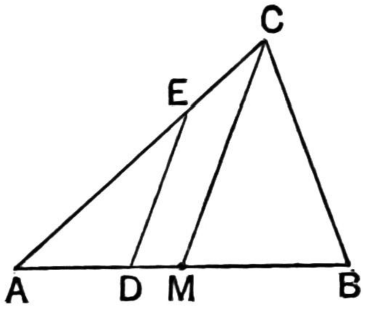




Ex. 426. In the annexed drawing $DE \parallel AB$, $DF \parallel CB$. Prove $AF : FB = BE : EC$.

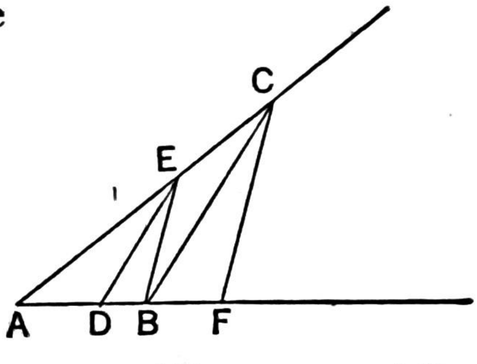
Ex. 427. A line is drawn through the intersection of the medians of a triangle, cutting two of the sides and parallel to the third side. In what ratio are these sides divided?

Ex. 428. In the drawing of Ex. 426, if $\frac{CD}{CA} = \frac{1}{4}$, prove that $DE = \frac{1}{4} AB$. What is the ratio of DF to CB?



Ex. 429. In the annexed drawing M is the midpoint of AB, $AD = \frac{1}{3} AB$, $DE \parallel CM$. What is the ratio of AE to AC?

Ex. 430. Given any angle and P a point within it. Draw a line through P meeting the sides of the angle in M and N, such that $MP = 2\,PN$.



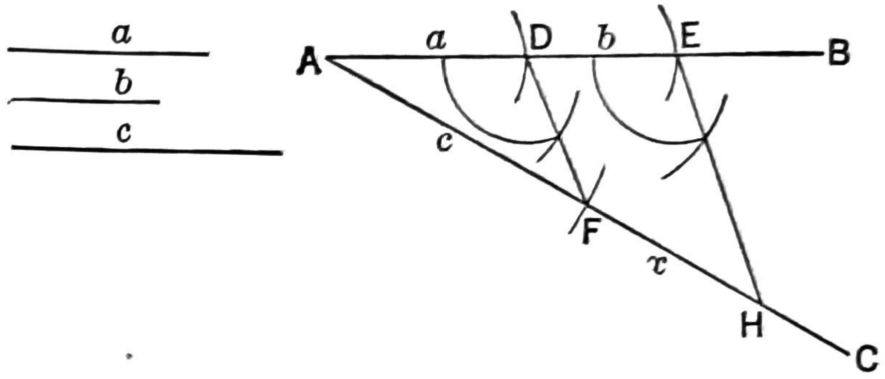
Ex. 431. In the annexed figure $DE \parallel BC$, $CF \parallel EB$. Prove $\frac{AD}{DB} = \frac{AB}{BF}$; $\frac{AD}{AB} = \frac{AB}{AF}$; $\frac{AB}{DB} = \frac{AF}{BF}$.

If $AD = 4''$, $AB = 6''$, find the value of BF and of AF.

What is the numerical value of the ratio $\frac{AE}{EC}$; also of $\frac{AC}{EC}$?

Proposition II. Problem

226. To construct the fourth proportional to three given lines.



Given lines a, b, and c.

Required to construct the fourth proportional to a, b, and c.

Analysis. If we let x equal the required line, then by definition of fourth proportional we have $a : b = c : x$. This suggests that we construct a $\triangle$ in which a, b, c, x are the segments of two sides made by a line $\parallel$ third side.

Construction. (*a*) Draw any convenient angle, such as BAC.

(*b*) On AB lay off $AD = a$, $DE = b$; on AC lay off $AF = c$.

(*c*) Draw DF, and through E draw a line $\parallel$ DF meeting AC at H. Then FH is the required line.

Proof. $a : b = c : x$ (by 222). Q.E.D.

227. The length of this fourth proportional is found by solving the equation for x. We then get $x = \frac{bc}{a}$. Hence Proposition II could have been stated as follows: Given three lines a, b, and c, find a line x such that its length shall be $\frac{bc}{a}$.

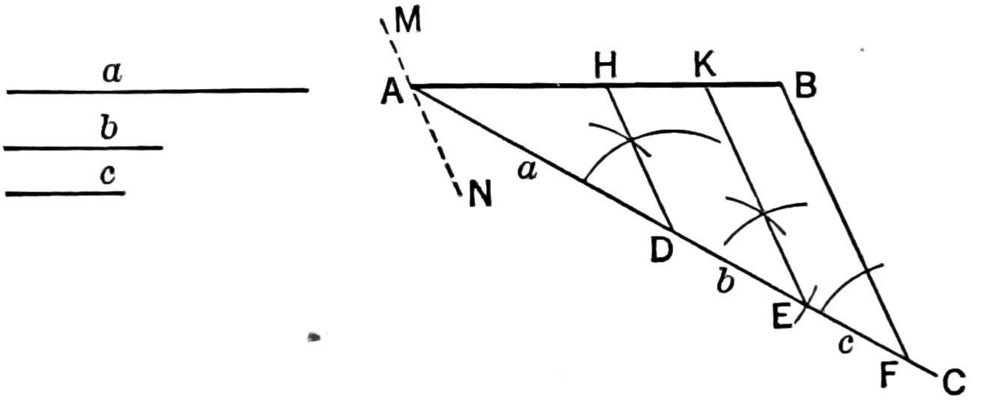
Ex. 432. If a, b, c are three given lines, construct a line whose length shall equal $\frac{ab}{c}$.

Analysis. Let $x = \frac{ab}{c}$, *then* $cx = ab$, i.e. $c : a = b : x$ (214). *This tells us that the required line is the fourth proportional to* c, a, b. *Now proceed with the construction.*

Ex. 433. If a and b are two given lines, construct a line whose length is $\frac{2a^2}{b}$; also one whose length is $\frac{a^2}{b}$.

Proposition III. Problem

228. To divide a given line into parts proportional to any number of given lines.



Given lines AB, a, b, c.

Required to divide AB into parts proportional to a, b, c.

Analysis. Since first part : second part : third part = $a : b : c$, *this suggests that we get a series of parallels cutting two transversals, and apply* 225.

Construction. (a) Draw AC making any convenient angle with AB.

(b) On AC lay off $AD = a$, $DE = b$, $EF = c$.

(c) Draw BF.

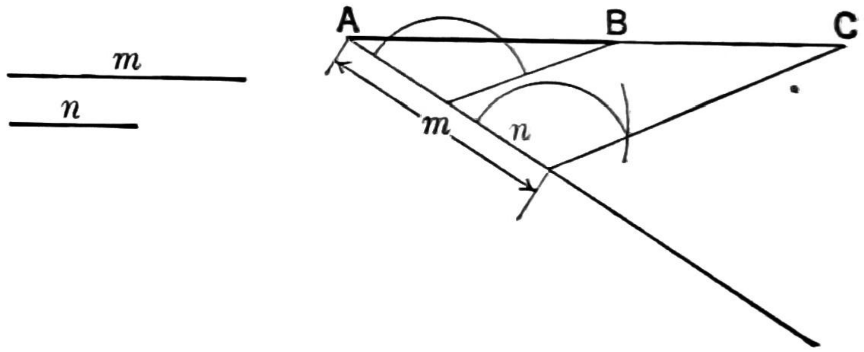
(*d*) Through E and D draw parallels to FB meeting AB in K and H respectively. Then AH, HK, and KB are the required parts.

Proof. Through A draw $MN \parallel BF$. $\therefore$ MN, HD, KE, and BF are parallel to each other (61). $\therefore AH : HK : KB = a : b : c$. (225) Q.E.D.

Ex. 434. Construct two lines when their sum and their ratio are given.

Sug. When a question does not state specifically what the given ratio is, we represent it by $m : n$, where m and n are treated as given lines.

Ex. 435. Construct two lines when their difference and their ratio are given.



Indicated solution. Hence AC and BC are the required lines.

If the difference of two lines is $5''$ and their ratio is $\frac{4}{3}$, find the length of each line.

Suppose their difference is k and their ratio $m : n$, what is the length of each line?

Ex. 436. Divide a line $12''$ long into parts which shall be to each other as $2 : 3 : 4$. Make a geometric construction and calculate the length of each part.

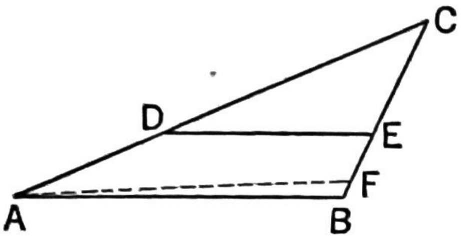
A line whose length is k is divided into three parts which are to each other as $a : b : c$.

Calculate the length of each part.

PROPORTIONAL SEGMENTS

Proposition IV. Theorem (*Converse of* 222)

229. If a line divides two sides of a triangle proportionally it is parallel to the third side.



Given $\triangle ABC$, with DE drawn so that $CD : DA = CE : EB$.

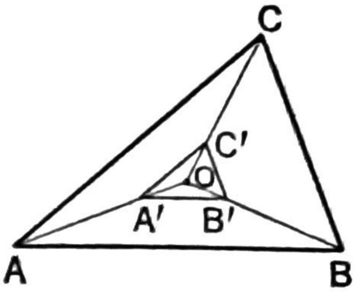
Prove $DE \parallel AB$.

Plan is to draw $AF \parallel DE$, *and then show that* AF *coincides with* AB.

Proof. Draw $AF \parallel DE$.

(1) $\frac{CD}{DA} = \frac{CE}{EB}$	(1) Given.
(2) but $\frac{CD}{DA} = \frac{CE}{EF}$	(2) 222.
(3) $\therefore \frac{CE}{EB} = \frac{CE}{EF}$	(3) Ax. 1.
(4) $\therefore EB = EF$	(4) Simplifying equation (3).
(5) $\therefore$ point F must be on point B	(5) $EB = EF$.
(6) $\therefore AF$ coincides with AB	(6) Only one straight line can be drawn between two points.
(7) $\therefore AB \parallel DE$	(7) AB coincides with a line which was drawn $\parallel DE$.

Q.E.D.

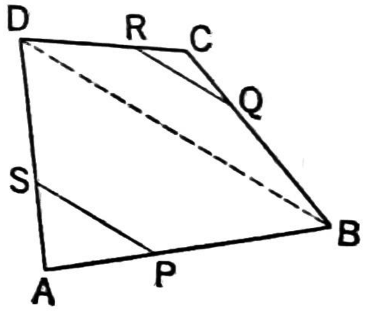


Ex. 437. In the annexed figure O is any point within $\triangle ABC$ and is joined to the vertices of the $\triangle$. Points A', B', C' are taken on OA, OB, OC respectively so that $A'B' \parallel AB$ and $B'C' \parallel BC$. Prove $A'C' \parallel AC$.

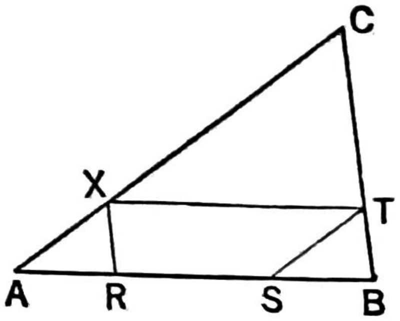
Ex. 438. Prove Ex. 437 if the point O is taken outside the $\triangle ABC$.

Ex. 439. Prove the first part of Prop. XXX, Book I, by using 229.

Ex. 440. P is a fixed point outside of a fixed line AB. Through P a variable line is drawn cutting AB at C. D is a point on PC so that $PD = \frac{1}{3} DC$. Find the locus of D as C moves along AB.



Ex. 441. If in the quadrilateral $ABCD$ the points P, Q, R, S are situated respectively on the sides AB, BC, CD, DA so that $AP : PB = AS : SD$ and $BQ : QC = DR : RC$, prove that $PS \parallel QR$.



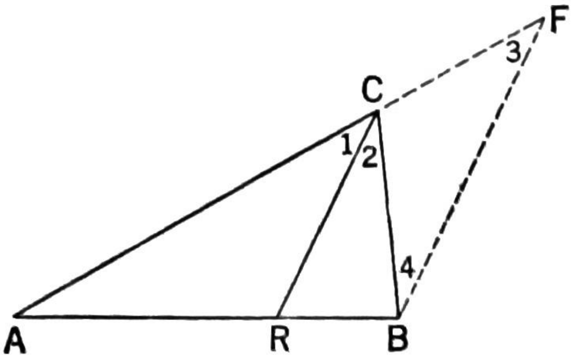
Ex. 442. In the annexed drawing $AR = SB$, $RX \parallel BC$, $ST \parallel AC$. Prove $XT \parallel AB$.

In this figure, if $AR = 2''$, $AB = 8''$, $CX = 5''$, find the length of ST, also of XT.

What is the value of the ratio $RX : BC$?

Proposition V. Theorem

230. The bisector of an angle of a triangle divides the opposite side into segments which are proportional to the other two sides.



Given △ ABC, with CR the bisector of ∠ ACB.

Prove $AR : RB = AC : BC$.

Analysis. The proportion suggests that we form a △ *where* CR *cuts two sides and is* ∥ *third side. This will enable us to obtain the ratio* $AR : RB$.

Proof. Through B draw a line ∥ CR, meeting AC prolonged at F. Then

(1) $AR : RB = AC : CF$	(1) 222.
(2) $\angle 1 = \angle 3$	(2) 58.
(3) $\angle 2 = \angle 4$	(3) 56.
(4) but $\angle 1 = \angle 2$	(4) Given.
(5) ∴ $\angle 3 = \angle 4$	(5) Ax. 1.
(6) ∴ $BC = CF$	(6) 73.
(7) ∴ $AR : RB = AC : BC$	(7) Substitution.

Q.E.D.

Note: For external division see 342, 343, 344.

Ex. 443. Show how to divide one side of a triangle in the ratio of the other two sides.

Ex. 444. The sides of a △ are 6″, 7″, 8″. The largest angle is bisected. Find the length of the segments of the opposite side.

Sug. Let the segments be x and $8 - x$.

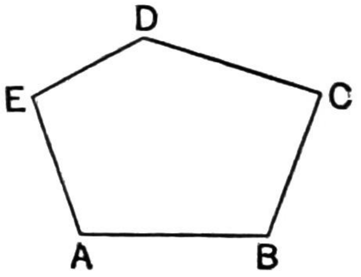
Ex. 445. State and prove the converse of Prop. V.

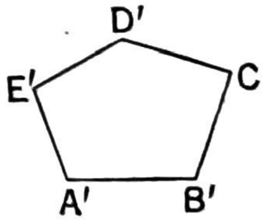
Sug. This can be proved by direct or indirect method. If direct, prolong AC and make $CF = CB$. If indirect, draw CR' so as to bisect $\angle ACB$, and then show that CR' and CR must coincide.

SIMILAR POLYGONS

231. Two polygons are *similar* if their corresponding angles are equal and their corresponding sides are proportional. Thus $ABCDE$ and $A'B'C'D'E'$ are similar if (1) $\angle A = \angle A'$, $\angle B = \angle B'$, $\angle C = \angle C'$, $\angle D = \angle D'$, $\angle E = \angle E'$, and (2)

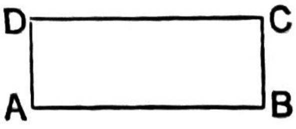
$$\frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{CD}{C'D'} = \frac{DE}{D'E'} = \frac{EA}{E'A'}.$$

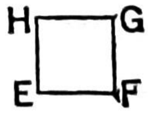


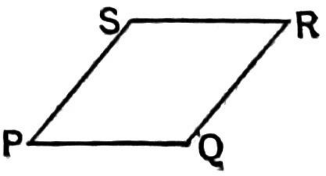


232. It is well to note that similar figures have the same *shape* but not necessarily the same size.

233. Note that similarity requires two conditions — the corresponding angles must be equal, and the corresponding sides must be proportional. Except in the case of triangles, one condition alone does not suffice. Thus the rectangle $ABCD$ and the square $EFGH$ have their corresponding angles equal, but the figures are not similar because the corresponding sides are not proportional. Again, the square $EFGH$ and the rhombus $PQRS$







have their corresponding sides proportional, but the corresponding angles are not equal. Hence the square and the rhombus are not similar.

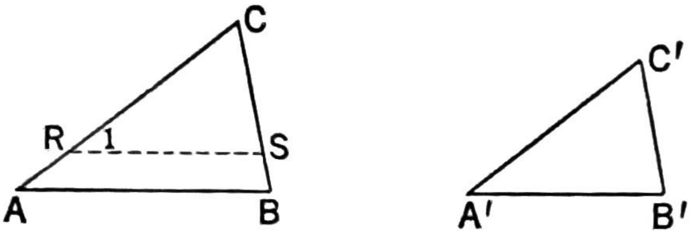
234. *Corresponding lines* of similar polygons are lines that are similarly situated with respect to the equal angles.

235. If two polygons have their angles respectively equal, the polygons are said to be *mutually equiangular.* If they have their sides respectively equal, they are said to be *mutually equilateral.*

SIMILAR TRIANGLES

Proposition VI. Theorem

236. Two mutually equiangular triangles are similar.



Given $\triangle$s ABC and $A'B'C'$ with $\angle A = \angle A'$, $\angle B = \angle B'$, $\angle C = \angle C'$.

Prove $\triangle ABC \sim \triangle A'B'C'$.

Analysis. It only remains to establish that the corresponding sides are proportional. This can be done by placing one $\triangle$ *upon the other, and we then have a line* $\|$ *one side of a* $\triangle$ *and cutting the other two sides.*

Proof. Place $\triangle A'B'C'$ upon $\triangle ABC$ so that $\angle C'$ coincides with its equal $\angle C$. Then $\triangle A'B'C'$ takes the position CRS.

(1) $\angle A' = \angle A$	(1) Given.
(2) $\therefore \angle 1 = \angle A$	(2) $\angle A'$ is now in the position $\angle 1$.
(3) $\therefore RS \parallel AB$	(3) 53.
(4) $\therefore AC : RC = BC : SC$	(4) 224.
(5) $\therefore AC : A'C' = BC : B'C'$	(5) Substitution.
(6) In like manner, by placing $\triangle A'B'C'$ upon $\triangle ABC$ so that $\angle B'$ coincides with $\angle B$, we obtain $AB : A'B' = BC : B'C'$	
(7) $\therefore AB : A'B' = BC : B'C' = AC : A'C'$	(7) Ax. 1.
(8) $\therefore \triangle ABC \sim \triangle A'B'C'$	(8) 231.

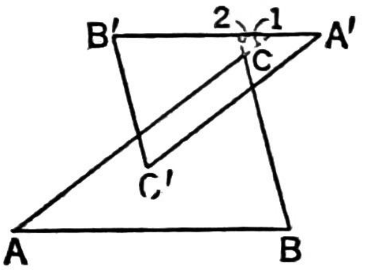
Q.E.D.

237. *Cor.* 1. Two triangles are similar if two angles of one are equal respectively to two angles of the other.

238. *Cor.* 2. Two right triangles are similar if an acute angle of one is equal to an acute angle of the other.

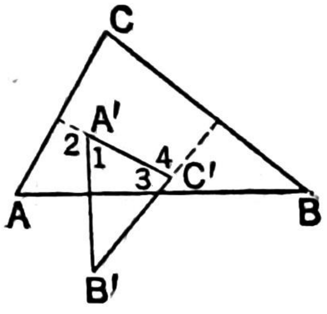
239. *Cor.* 3. Two triangles which have their sides respectively parallel are similar.

Outline of proof: $\angle A = \angle 1 = \angle A'$; $\angle B = \angle 2 = \angle B'$



240. *Cor.* 4. Two triangles which have their sides respectively perpendicular are similar.

Outline of proof: $\angle 1 + \angle 2 = 180°$, but $\angle A + \angle 2 = 180°$. $\therefore \angle A = \angle 1$. In like manner $\angle C = \angle 3$ (each is the sup. of $\angle 4$).



Ex. 446. Prove that a line parallel to the base of a triangle, intersecting the other two sides, forms a triangle similar to the given triangle.

Ex. 447. Two isosceles triangles are similar if any angle of one equals the corresponding angle of the other.

Ex. 448. Prove that any two equilateral triangles are similar.

Ex. 449. The sides of a triangle are $5''$, $6''$, $8''$. Find the sides of a similar triangle if the side corresponding to 8 is $12''$.

Ex. 450. If the altitudes AD and BE of the triangle ABC intersect in F, prove that $\triangle AFE$ is similar to $\triangle BFD$, and that $\triangle ACD$ is similar to $\triangle BCE$.

Ex. 451. If two chords AB and CD intersect within the circle at the point F, prove that $\triangle AFC \sim \triangle BFD$.

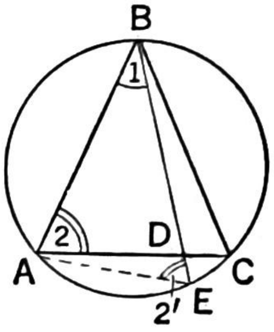
Ex. 452. ABC is a $\triangle$ inscribed in a $\odot$, and at vertex C a tangent is drawn. A line parallel to this tangent cuts AC at D and BC at E. Prove that $\triangle DEC \sim \triangle ABC$.

Ex. 453. Prove that two polygons similar to a third polygon are similar to each other.

241. *We can prove four lines proportional if we can prove that they are corresponding sides of similar triangles.*

Let the pupil study carefully the following example:

Ex. 454. From the vertex B of the inscribed isosceles $\triangle ABC$ a chord BE is drawn cutting the base in D. Prove that $BE : AB = AB : BD$.



Analysis. The proportion suggests that we look for $\sim$ △s. *BE and AB are sides of $\triangle ABE$, AB and BD are sides of $\triangle ABD$. Hence we must draw AE, and then prove $\triangle ABE \sim \triangle ABD$.*

Proof. Draw AE. Then in the $\triangle$s ABE and ABD

(1) $\angle 1 = \angle 1$	(1) Iden.
(2) $\angle 2 = \angle C$	(2) 42.
(3) but $\angle C = \angle 2'$	(3) 178.
(4) $\therefore \angle 2 = \angle 2'$	(4) Ax. 1.
(5) $\therefore \triangle ABE \sim \triangle ABD$	(5) 237.
(6) $\therefore BE : AB = AB : BD$	(6) Corr. sides of $\sim$ polygons are in proportion.

Q.E.D.

242. *Note:* To form the proportion we take two sides of one triangle and compare them with the corresponding sides of the other triangle. Thus $BE : AB = ? : ?$ We reason as follows: BE is opposite $\angle BAE$. The equal of $\angle BAE$ is $\angle ADB$. Opposite $\angle ADB$ is AB. $\therefore AB$ is the third term of the proportion. To obtain the fourth term we study the second term. AB is opposite $\angle 2'$. The equal of $\angle 2'$ is $\angle 2$. Opposite $\angle 2$ is BD. Hence BD is the fourth term of the proportion. Note that in the case of similar triangles, corresponding sides are opposite the equal angles.

Ex. 455. Prove that the diagonals of a trapezoid divide each other proportionally.

Ex. 456. The corresponding altitudes of two similar triangles are to each other as any two corresponding sides.

Ex. 457. If in the right triangle ABC, right-angled at C, CD is drawn $\perp AB$, prove $AB : AC = AC : AD$; also prove $AB : BC = BC : DB$.

Ex. 458. The bisector of $\angle C$ of the inscribed $\triangle ABC$ cuts AB in D and the circumference in E. Prove that $AC : CD = CE : BC$.

Ex. 459. AB is a diameter of $\odot O$ and BC is a tangent to this $\odot$. AC meets the $\odot$ at D. Prove AB is the mean proportional between AC and AD.

Ex. 460. ABC is a triangle in which $\angle A = \angle B = 2 \angle C$. Prove that if the bisector of $\angle A$ meets the opposite side at D, AB is the mean proportional between BD and BC.

Ex. 461. Two circles are tangent internally at P. PA and PB are chords of the larger ⊙ cutting the smaller ⊙ in C and D respectively. Prove $PA : PC = PB : PD$.

243. *We can prove the product of two lines equal to the product of two other lines, if we can prove that the four lines are in proportion.*

Ex. 462. If P is any point in chord AB, and $PC \perp$ diameter AD, prove that $AC \times AD = AB \times AP$.

Analysis. We can prove the required equation if we can prove $AC : AP = AB : AD$. We can prove this proportion if we can prove $\triangle ACP \sim \triangle ABD$.

Ex. 463. In any right triangle the product of the legs equals the product of the hypotenuse and the altitude upon the hypotenuse.

Ex. 464. In a ⊙ two chords AB and CD intersect at E, C being the midpoint of minor arc AB. Draw AC and prove that $\overline{AC}^2 = CE \times CD$.

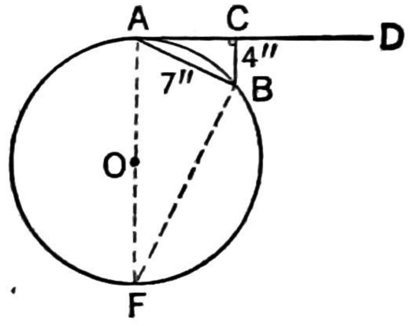
Ex. 465. P is an external point to a ⊙. PAB and PCD are secants cutting the ⊙ in points A, B, and C, D respectively. Prove that $PA \times PB = PC \times PD$.

Ex. 466. The product of any side of a triangle and its altitude is equal to the product of any other side and its altitude.

Ex. 467. If from a point external to a ⊙ a tangent and a secant are drawn, the product of the whole secant and its external segment is equal to the square of the tangent. (See 249.)

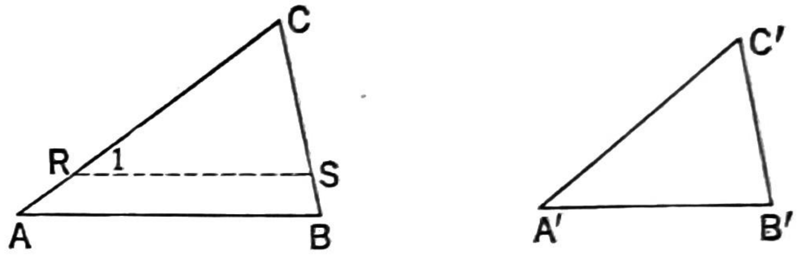
Ex. 468. In the annexed drawing AD is tangent to ⊙ O at A. AB is a chord 7″ long. $BC \perp AD$, $BC = 4''$. Find the radius.

Sug. Make use of the $\sim$ ▲ ABF and ABC.



Proposition VII. Theorem

244. If two triangles have an angle of one equal to an angle of the other and the including sides proportional, the triangles are similar.



Given $\triangle$s ABC and $A'B'C'$ with $\angle C = \angle C'$ and $AC : A'C' = BC : B'C'$.

Prove $\triangle ABC \sim \triangle A'B'C'$.

Analysis. If we can get another pair of angles equal, then the $\triangle$s *will be* $\sim$ *by* 237. *This we can do by placing one* $\triangle$ *upon the other, and thus get* $RS \parallel AB$.

Proof. Place $\triangle A'B'C'$ upon $\triangle ABC$ so that $\angle C'$ coincides with its equal $\angle C$. Then $A'B'$ takes the position RS. Hence

(1) $AC : A'C' = BC : B'C'$	(1) Given.
(2) $\therefore AC : RC = BC : SC$	(2) Sub.
(3) $\therefore AC - RC : RC = BC - SC : SC$	(3) 218.
(4) $\therefore AR : RC = BS : SC$	(4) Sub.
(5) $\therefore RS \parallel AB$	(5) 229.
(6) $\therefore \angle 1 = \angle A$	(6) 58.
(7) $\therefore \angle A' = \angle A$	(7) Sub.
(8) but $\angle C = \angle C'$	(8) Given
(9) $\therefore \triangle ABC \sim \triangle A'B'C'$	(9) 237. Q.E.D.

Ex. 469. If the legs of one right $\triangle$ are proportional to the legs of another right $\triangle$, the $\triangle$s are similar.

Ex. 470. The corresponding medians of two similar △s are to each other as any two corresponding sides.

Ex. 471. AD and BE are two altitudes of $\triangle ABC$. Draw DE and prove that $\triangle ABC \sim \triangle CED$.

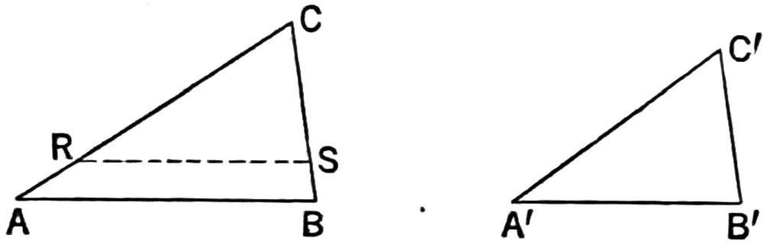
Sug. First prove $\triangle ADC \sim \triangle BEC$, and then apply 244.

Ex. 472. The radii of the circumscribed circles of two similar triangles are to each other as any two corresponding sides.

Sug. Prove corresponding central ∠s equal, and then apply 244.

Proposition VIII. Theorem

245. If two triangles have their sides respectively proportional, the triangles are similar.



Given △s ABC and $A'B'C'$ with $AB : A'B' = BC : B'C' = AC : A'C'$.

Prove $\triangle ABC \sim \triangle A'B'C'$.

Analysis. If we can construct a △ on ABC and prove this constructed △ ∼ △ ABC and congruent to △ A′B′C′, then the theorem is established.

Proof. On CA lay off $CR = C'A'$, on CB lay off $CS = C'B'$, and draw RS. (Why cannot we place $\triangle A'B'C'$ upon $\triangle ABC$?)

(1) $AC : A'C' = BC : B'C'$	(1) Given.
(2) $\therefore AC : RC = BC : SC$	(2) Sub.
(3) $\therefore \triangle ABC \sim \triangle RSC$	(3) 244.

(4) $\therefore AC : RC = AB : RS$	(4) Corr. sides of $\sim$ $\triangle$ are in proportion.
(5) $\therefore AC : A'C' = AB : RS$	(5) Sub.
(6) but $AC : A'C' = AB : A'B'$	(6) Given.
(7) $\therefore AB : RS = AB : A'B'$	(7) Ax. 1.
(8) $\therefore RS \times AB = AB \times A'B'$	(8) 213.
(9) $\therefore RS = A'B'$	(9) Ax. 5.
(10) $\therefore \triangle RSC \cong \triangle A'B'C'$	(10) 46.
(11) $\therefore \triangle ABC \sim \triangle A'B'C'$	(11) Sub. in step (3).

Q.E.D.

Note: For trigonometry of the right triangle see 357–366.

Exercises

Ex. 473. The sides of one $\triangle$ are 4″, 6″, 8″. The sides of another $\triangle$ are 6″, 9″, 12″. Are the $\triangle$ similar? What is the ratio of two corresponding altitudes? of two corresponding medians?

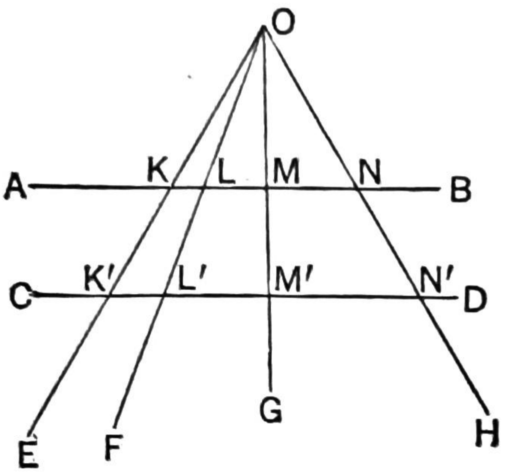
Ex. 474. If two sides and the median to one of these sides in one triangle are proportional to the corresponding parts of another triangle, prove that the two triangles are similar.

Ex. 475. Two right triangles are similar if the hypotenuse and a leg of one triangle are proportional to the hypotenuse and leg of the other.

Sug. Draw the median on each hypotenuse, and then apply Ex. 100, 245, 238.

246. *If we have to prove four lines in proportion, and these lines are not sides of two similar triangles, we aim to prove each of the given ratios equal to a third ratio.*

Ex. 476. If two parallels are cut by three or more transversals passing through a common point, the corresponding segments of the parallels are in proportion.



Outline of proof:

$$\frac{KL}{K'L'}=\left(\frac{OL}{OL'}\right)=\frac{LM}{L'M'}=\left(\frac{OM}{OM'}\right)=\frac{MN}{M'N'}$$

Note: The parenthesis about a ratio indicates that this ratio is used to prove the equality of the ratios immediately preceding and following it.

Ex. 477. Prove the above example when the point O is between the parallels.

Ex. 478. $\triangle ABC \sim \triangle A'B'C'$. AD bisects $\angle A$, $A'D'$ bisects $\angle A'$. Prove $AD : A'D' = BC : B'C'$.

Ex. 479. Prove that in any triangle the median to one side bisects all lines parallel to that side and terminated by the other two sides.

Ex. 480. In parallelogram $ABCD$ a line DE is drawn meeting the diagonal AC in F, the side BC in G, and the side AB produced in E. Prove $\overline{DF}^2 = FG \times FE$.

GROUP XI

Complete each of the following statements:

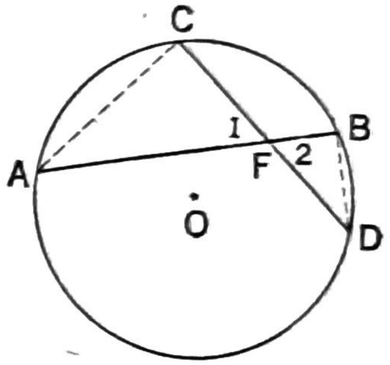
(*a*) The vertex angle of isosceles $\triangle ABC$ is 40°. A base angle of isosceles $\triangle DEF$ is 70°. $\triangle$s ABC and DEF are . . . .

(*b*) Two sides of a triangle are $7''$ and $10''$. The bisector of their included angle divides the third side into segments that are in the ratio . . . .

(*c*) The corresponding altitudes of two similar triangles are in the ratio $4 : 5$. If a median of the first triangle is $9''$, the corresponding median of the second triangle is . . . .

Proposition IX. Theorem

247. If two chords intersect within a circle, the product of the segments of one is equal to the product of the segments of the other.



Given $\odot O$ and chords AB and CD intersecting in F.

Prove $AF \times FB = CF \times FD$.

Analysis. The required equation suggests the proportion $AF : CF = FD : FB$, *which in turn suggests the* $\sim$ $\triangle$s AFC *and* BFD.

Proof. Draw AC and BD. Then

(1) $\angle A$ is measured by $\frac{1}{2}\overset{\frown}{BC}$	(1) 176.
(2) $\angle D$ is measured by $\frac{1}{2}\overset{\frown}{BC}$	(2) 176.
(3) $\therefore \angle A = \angle D$	(3) Ax. 1.
(4) $\angle 1 = \angle 2$	(4) 28.
(5) $\therefore \triangle AFC \sim \triangle BFD$	(5) 237.
(6) $\therefore AF : CF = FD : FB$	(6) Corr. sides of $\sim$ $\triangle$s are in proportion.
(7) $\therefore AF \times FB = CF \times FD$	(7) 213.

Q.E.D.

248. *Cor.* If through a fixed point within a circle any chord is drawn, the product of its segments is constant.

Sug. Draw the diameter through the fixed point.

Ex. 481. In the drawing for Prop. IX suppose $AF = 5''$, $FB = 2''$, $CF = 3''$. Find the length of FD.

Ex. 482. ⊙ O has a $6''$ radius. P is a point $2''$ from the center of the ⊙. Any chord is drawn through P. What is the product of its segments?

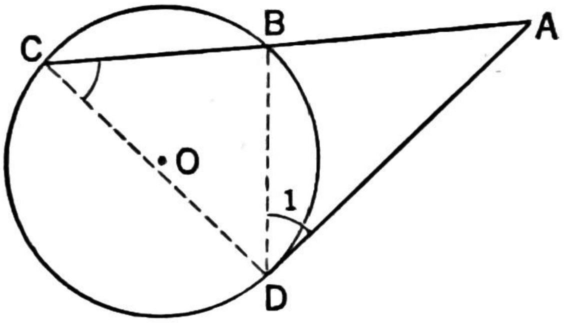
Ex. 483. AB is a chord of a ⊙. Through the midpoint of AB another chord is drawn such that its segments are $4''$ and $2''$. Find the length of AB.

249. A secant from an external point to a circle is the segment of the secant which lies between the given point and the second point of intersection of the secant and the circle. Thus in Prop. X AC is the secant from A.

SECANT AND TANGENT

Proposition X. Theorem

250. If from a point outside a circle a secant and a tangent are drawn, the tangent is the mean proportional between the secant and its external segment.



Given AD tangent to ⊙ O at D, and secant AC cutting the ⊙ at B and C.

Prove $AC : AD = AD : AB$.

Analysis. The required proportion suggests that we get the $\sim$ *△ ACD and ABD.*

Proof. Draw CD and BD, thus forming the $\triangle ACD$ and ABD. Then

(1) $\angle A = \angle A$	(1) Iden.
(2) $\angle C$ is measured by $\frac{1}{2}\overset{\frown}{BD}$	(2) 176.
(3) $\angle 1$ is measured by $\frac{1}{2}\overset{\frown}{BD}$	(3) 181.
(4) $\therefore \angle C = \angle 1$	(4) Ax. 1.
(5) $\therefore \triangle ACD \sim \triangle ABD$	(5) 237.
(6) $\therefore AC : AD = AD : AB$	(6) Corr. sides of $\sim$ $\triangle$ are in proportion.

Q.E.D.

251. *Cor.* If from a fixed point outside a circle a variable secant is drawn, the product of the secant and its external segment is constant.

Sug. The length of a tangent from that point is constant.

Exercises

Ex. 484. Two ⓢ are tangent internally, and from any point in the common tangent secants are drawn to the two ⓢ. Prove that the product of the secants and their external segments are equal.

Ex. 485. The tangents to two intersecting ⓢ drawn from any point in their common chord produced are equal.

Ex. 486. The common chord of two intersecting ⓢ, if produced, will bisect their common tangents.

Ex. 487. Two ⓢ intersect, and from any point in the prolongation of the common chord secants are drawn to the two ⓢ. Prove that the products of the secants and their external segments are equal.

Ex. 488. P is a point 7″ from the center of a ⊙. Through P a secant is drawn such that its internal segment is 5″ and its external segment is 3″. Find the radius of the ⊙.

Ex. 489. One of two secants meeting outside a ⊙ is $12\frac{1}{2}$″ and its external segment is 4″. The other secant is divided into two equal parts by the ⊙. Find the length of the second secant.

Ex. 490. How far from a ⊙ whose radius is 6″ must a point be taken so that the secant from that point through the center of the ⊙ shall be twice the tangent from that point?

Ex. 491. The radius of a ⊙ is 2″. From a point 4″ from the center a secant is drawn so that its internal segment is 1″. Find the length of the secant.

Ex. 492. How far could one see from an aeroplane that is one mile above the surface of the earth?

Sug. Assume the earth is a sphere 8000 miles in diameter.

RIGHT TRIANGLE

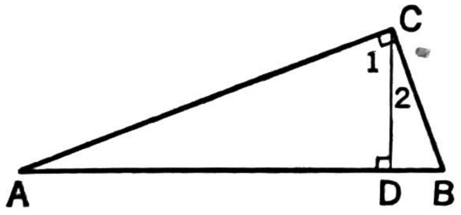
Proposition XI. Theorem

252. If in a right triangle the altitude upon the hypotenuse is drawn, then

I. The two triangles thus formed are similar to the given triangle and to each other.

II. The altitude is the mean proportional between the segments of the hypotenuse.

III. Each leg of the given triangle is the mean proportional between the hypotenuse and the segment adjacent to that leg.



Given $\triangle ABC$ with right angle at C and $CD \perp AB$.

Prove I. $\triangle ADC \sim \triangle ABC$, $\triangle BDC \sim \triangle ABC$,
$\triangle ADC \sim \triangle BDC$.

II. $AD : CD = CD : DB$.

III. $AB : AC = AC : AD$ and $AB : BC = BC : DB$.

Analysis. We can prove I if we can get an acute $\angle$ *of one right* $\triangle$ *equal to an acute* $\angle$ *of the other. We can prove II and III by* $\sim$ $\triangle$s.

Proof. I. In the right $\triangle$s ADC and ABC

(1) $\angle A = \angle A$	(1) Iden.
(2) $\therefore \triangle ADC \sim \triangle ABC$	(2) 238.

In like manner $\triangle BDC$ can be proved $\sim \triangle ABC$. In right $\triangle$s ADC and BDC

(3) $\angle A = 90° - \angle 1$	(3) 66.
(4) $\angle 2 = 90° - \angle 1$	(4) 14.
(5) $\therefore \angle A = \angle 2$	(5) Ax. 1.
(6) $\therefore \triangle ADC \sim \triangle BDC$	(6) 238.
II. $AD : CD = CD : DB$	Corr. sides of $\sim$ $\triangle$s are in proportion.
III. $AB : AC = AC : AD$	Corr. sides of $\sim$ $\triangle$s are in proportion.
$AB : BC = BC : DB$	Corr. sides of $\sim$ $\triangle$s are in proportion.

Q.E.D.

253. *Cor.* The perpendicular from any point on a circle to a diameter is the mean proportional between the segments of the diameter.

Ex. 493. The diameter of a ⊙ is 24″ and is divided into three equal parts. Find the length of the chord drawn through one of the points of division perpendicular to the diameter.

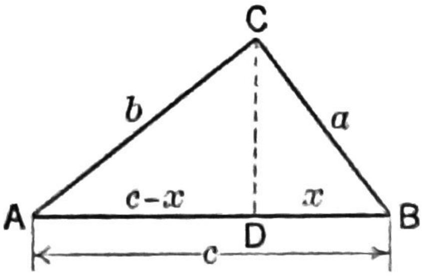
Ex. 494. The altitude upon the hypotenuse of a right triangle is 12′ and the greater segment of the hypotenuse is 16′. Find the length of each side of the △.

Ex. 495. A chord 24″ long is 9″ from the center of the ⊙. Find the length of the tangents drawn from the extremities of the chord and produced till they meet.

Ex. 496. If the altitude to the hypotenuse of a right triangle is drawn, prove that the segments of the hypotenuse are to each other as the squares of the adjacent legs of the triangle.

Proposition XII. Theorem

254. The square of the hypotenuse of a right triangle is equal to the sum of the squares of the legs.



Given right △ ABC, hypotenuse denoted by c, and legs by a and b.

Prove $c^2 = a^2 + b^2$.

Analysis. We can prove the required equation if we can find a^2 and b^2 and then add the results. a^2 and b^2 can be found by 252 *III.*

Proof. Draw $CD \perp AB$. Denote DB by x, then $AD = c - x$.

(1) $c : a = a : x$ and $c : b = b : c - x$	(1) 252 III.
(2) $\therefore a^2 = cx$ and $b^2 = c^2 - cx$	(2) 213.
(3) $\therefore a^2 + b^2 = c^2$ or $c^2 = a^2 + b^2$	(3) Ax. 2.
	Q.E.D.

255. *Note:* This proposition is perhaps the most celebrated and important proposition in geometry. It is known as the Pythagorean Theorem, because it is supposed to have been first proved by Pythagoras, a famous Greek mathematician, about 525 B.C. The proof given by Pythagoras is not known.

256. *Cor.* The square of either leg of a right triangle equals the square of the hypotenuse minus the square of the other leg.

Exercises

Ex. 497. The side of a square is 5″. Find the length of the diagonal.

Ex. 498. A side of an equilateral △ is 4″. Find the altitude.

Ex. 499. A side of an equilateral △ is b. Find the altitude.

Ex. 500. The altitude of an equilateral △ is h. Find the side.

Ex. 501. Find the length of the longest chord and of the shortest chord that can be drawn through a point 5′ from the center of a ⊙ whose radius is 9′.

Ex. 502. Two parallel chords of a ⊙ are each 18″ and the distance between them is 10″. Find the radius of the ⊙.

Ex. 503. The legs of a right △ are 8″ and 9″ respectively. Find the radius of the circumscribed ⊙.

Ex. 504. The radius of a ⊙ is 12″. Find the length of a tangent drawn from a point 13″ from the center of the ⊙.

Ex. 505. The radii of two concentric Ⓢ are 9″ and 15″ respectively. Find the length of a chord of the outer ⊙ which is tangent to the inner ⊙.

Ex. 506. Find the altitude of an isosceles trapezoid if the parallel sides are 40″ and 58″ respectively and each of the other sides is 41″.

Ex. 507. The legs of a right △ are 5′ and 12′ respectively. How long are the segments of the hypotenuse made by the bisector of the right angle?

Ex. 508. A chord 1′ long is 4″ from the center of the ⊙. How far from the center is a chord 9″ long?

Ex. 509. A chord 20″ long subtends an arc of 120°. Find the distance of the chord from the center of the ⊙.

Ex. 510. In a ⊙ whose radius is 17″, an arc is subtended by a chord 30″ long. Find the length of the chord of half this arc.

Ex. 511. The chord of an arc is 42′. The chord of half that arc is 29′. Find the diameter of the ⊙.

Ex. 512. One angle of a rhombus is 60°. The longer diagonal is 14″. Find the side.

Ex. 513. B and C are the extremities of an arc of 120° on a ⊙ whose radius is 2″. Tangents at B and C meet at A. Find the perimeter of $\triangle ABC$.

Ex. 514. Two parallel chords in a ⊙ are one inch apart. Find the radius of the ⊙ if the chords are 8″ and 6″ respectively.

Ex. 515. Two Ⓢ have radii of 8″ and 3″ respectively, and the distance between their centers is 15″. Find the length of the common external tangent; also of the common internal tangent.

Ex. 516. The perimeter of a rectangle is 14′ and its diagonal is 5′. Find the sides of the rectangle.

Ex. 517. Two Ⓢ whose radii are 10″ and 6″ respectively intersect in such a way that their common chord is 5″. Find the distance between their centers.

Ex. 518. Two tangents each 24″ long are drawn from the same point to a ⊙ of radius 7″. Find the length of the chord of contact.

Ex. 519. $ABCD$ is a rhombus whose diagonals intersect at O. Prove $OA \times OC = \overline{AB}^2 - \overline{OB}^2$.

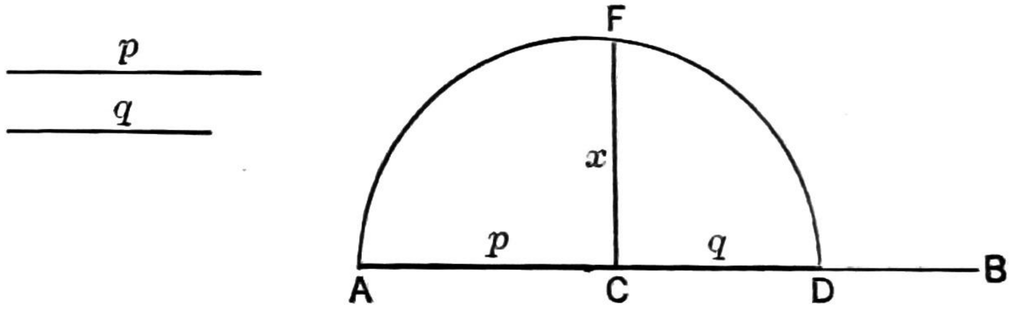
Ex. 520. Choose two points A and B upon a given straight line, and two other points C and D upon a straight line $\perp AB$. Prove that the hypotenuse of a right $\triangle$ whose legs are equal to AC and BD is equal to the hypotenuse of a right △ whose legs are equal to AD and BC.

Sug. Prove $\overline{AC}^2 + \overline{BD}^2 = \overline{AD}^2 + \overline{BC}^2$ by writing down the expression for each, and then adding.

MEAN PROPORTIONAL

Proposition XIII. Problem

257. To construct the mean proportional between two given lines.



Given lines p and q.

Required to construct the mean proportional between p and q.

Analysis. If we assume the problem done, and denote the required line by x, then by definition of mean proportional we have $p : x = x : q$. 253 suggests that x can be made a perpendicular upon diameter $p + q$.

Construction. (*a*) On an indefinite line AB lay off $AC = p$, $CD = q$.

(*b*) On AD as diameter draw a semicircle.

(*c*) At C erect a perpendicular to AD, and let it meet the ⊙ at F. Then CF is the required line.

Proof. Denote CF by x. Then $p : x = x : q$ (by 253).

258. The length of this mean proportional is found by solving the above equation. Thus, $x^2 = pq$. $\therefore x = \sqrt{pq}$. Hence 257 might have been stated as follows: Given two lines p and q, required to construct a line x such that $x = \sqrt{pq}$.

Ex. 521. If a and b are two given lines, construct a line whose length shall be $\sqrt{3\,ab}$.

Analysis. Let $x = \sqrt{3\,ab}$, then $x^2 = 3\,ab$, $\therefore 3\,a : x = x : b$

Ex. 522. If m is a given line, construct $m\sqrt{5}$. Solve by two different methods.

Ex. 523. Construct a line equal to $\sqrt{6}''$.

GROUP XII

Complete each of the following statements:

(*a*) A chord 12″ long is 5″ from the center of the circle. The radius is . . . .

(*b*) The diagonal of a square is $3\sqrt{2}$. Its side is . . . .

(*c*) The altitude of an equilateral triangle is 10″. Its side is . . . .

(*d*) From a point external to a circle, a secant is drawn whose external segment equals the internal segment. If the length of the tangent is 8″, the length of the secant is . . . .

(*e*) The altitude upon the hypotenuse of a right triangle divides the hypotenuse into segments that are 4″ and 9″. The altitude is . . . , and the perimeter is . . . .

(*f*) Two circles whose radii are 2″ and 4″ are tangent externally. Their common external tangent is . . . .

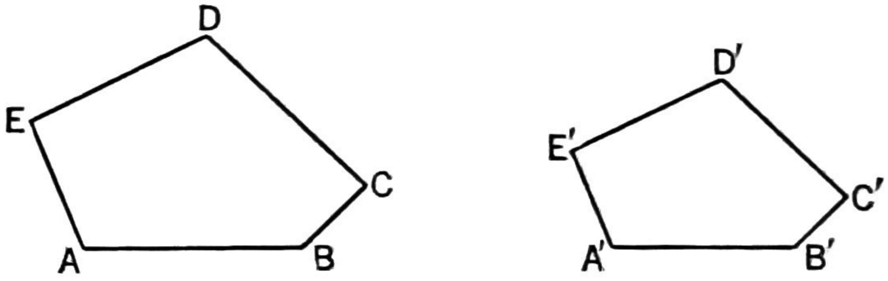
(*g*) Three circles whose radii are 3″, 2″, and 1″ respectively are so situated that each is tangent to the other two. If the centers of the circles are connected by straight lines, the triangle thus formed is . . . .

(*h*) If a square is inscribed in a semicircle of radius r, a side of the square is . . . .

SIMILAR POLYGONS

Proposition XIV. Theorem

259. The perimeters of two similar polygons are to each other as any two corresponding sides.



Given polygon $ABCDE$ similar to polygon $A'B'C'D'E'$, with their respective perimeters denoted by p and p'.

Prove $p : p' = AB : A'B'$.

Analysis. The corresponding sides are in proportion. We thus get a series of equal ratios, and if we apply 219 *we obtain the value of the ratio* $p : p'$.

Proof.

(1) $\dfrac{AB}{A'B'} = \dfrac{BC}{B'C'} = \dfrac{CD}{C'D'} = \dfrac{DE}{D'E'} = \dfrac{EA}{E'A'}$ | (1) Corr. sides of ∼ polygons are in proportion.

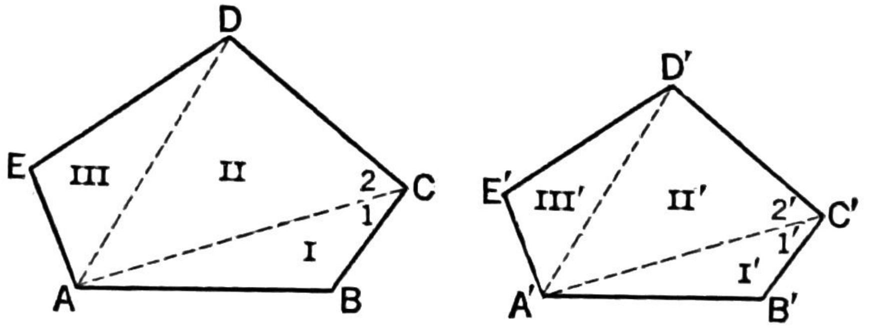
(2) $\therefore \dfrac{AB + BC + CD + DE + EA}{A'B' + B'C' + C'D' + D'E' + E'A'} = \dfrac{AB}{A'B'}$ | (2) 219.

(3) $\therefore \dfrac{p}{p'} = \dfrac{AB}{A'B'}$ | (3) Sub.

Q.E.D.

Proposition XV. Theorem

260. Two similar polygons may be divided into the same number of triangles, similar each to each, and similarly placed.



Given polygon $ABCDE \sim$ polygon $A'B'C'D'E'$.

Prove $ABCDE$ and $A'B'C'D'E'$ can be divided into the same number of triangles, similar each to each, and similarly placed.

Plan is to draw diagonals from corresponding vertices, and then prove the corresponding $\triangle$s *similar by* 244.

Proof. From two corresponding vertices A and A' draw the diagonals AC, AD, $A'C'$, $A'D'$, thus dividing each polygon into three $\triangle$s similarly placed. (In general, into $n - 2$ triangles.)

(1) since $\angle B = \angle B'$	(1) Corr. $\angle$s of $\sim$ polygons are equal.
(2) and $AB : A'B' = BC : B'C'$	(2) Corr. sides of $\sim$ polygons are in proportion.
(3) $\therefore \triangle I \sim \triangle I'$	(3) 244.

(4) In like manner $\triangle III$ can be proved $\sim \triangle III'$. It only remains to prove $\triangle II \sim \triangle II'$.

(5) $\angle 1 = \angle 1'$	(5) Corr. $\angle$s of $\sim$ $\triangle$s are equal.

(6) and $\angle BCD = \angle B'C'D'$	(6) Corr. ∡ of $\sim$ polygons are equal.
(7) $\therefore \angle 2 = \angle 2'$	(7) Ax. 3.
(8) Now $\frac{BC}{B'C'} = \frac{AC}{A'C'}$	(8) Corr. sides of $\sim$ △ are in proportion.
(9) and $\frac{BC}{B'C'} = \frac{CD}{C'D'}$	(9) Corr. sides of $\sim$ polygons are in proportion.
(10) $\therefore \frac{AC}{A'C'} = \frac{CD}{C'D'}$	(10) Ax. 1.
(11) $\therefore \triangle II \sim \triangle II'$	(11) 244.

Q.E.D.

Ex. 524. The sides of a quadrilateral are 2″, 3″, 5″, 8″. The longest side of a similar quadrilateral is 9″. Find its perimeter.

Ex. 525. The perimeters of two similar triangles are to each other as any two corresponding medians.

Ex. 526. The perimeters of two similar polygons are to each other as any two corresponding diagonals.

GROUP XIII

State whether the following statements are true or false, and give reasons.

(*a*) All congruent figures are similar.

(*b*) Any two rectangles are similar.

(*c*) To prove two polygons similar, it is sufficient to prove that their corresponding sides are in proportion.

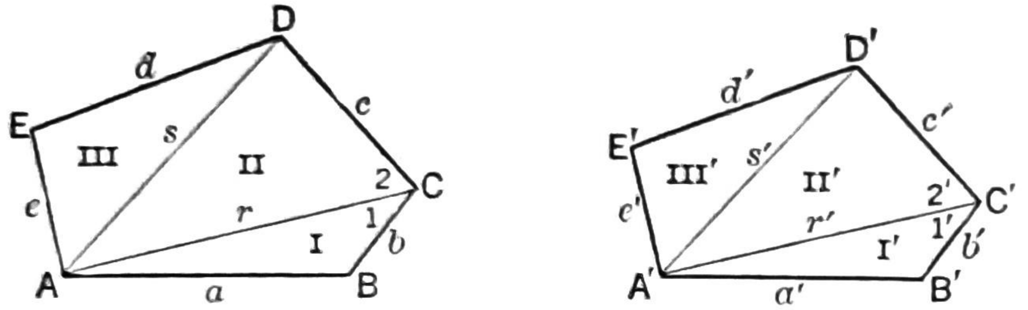
(*d*) The bisector of an angle of any triangle bisects the opposite side.

(*e*) Given a unit length, it is possible to construct accurately a line whose length is $\sqrt{3}$.

(*f*) The base of a triangle is 9″ and its altitude is 5″. If a square is inscribed in the triangle, its side is $3\frac{3}{11}''$.

Proposition XVI. Theorem. (*Converse of Prop. XV*)

261. If two polygons are composed of the same number of triangles, similar each to each and similarly placed, then the polygons are similar.



Given polygon $ABCDE$ composed of $\triangle$s I, II, III and polygon $A'B'C'D'E'$ composed of $\triangle$s I', II', III'; also $\triangle I \sim \triangle I'$, $\triangle II \sim \triangle II'$, $\triangle III \sim \triangle III'$; and that the triangles are similarly placed.

Prove polygon $ABCDE \sim$ polygon $A'B'C'D'E'$.

Analysis. If we can prove the corresponding angles equal and the corresponding sides proportional, the theorem is established. This we can do by studying pairs of $\sim$ $\triangle$s.

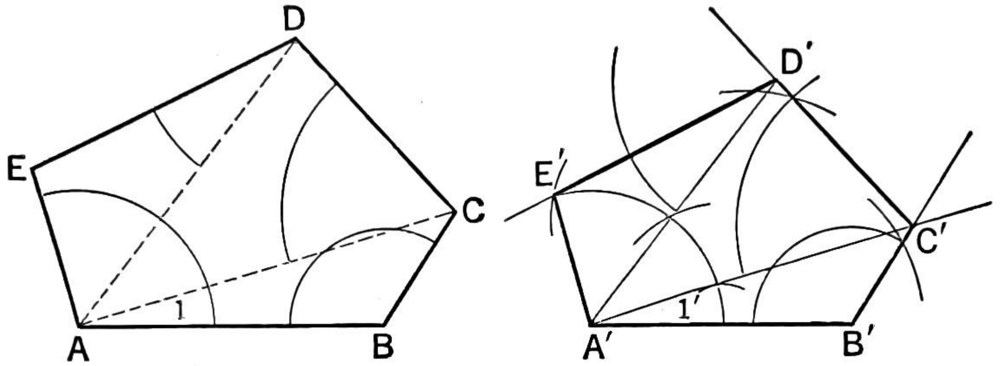
Proof.

(1) $\angle B = \angle B'$ and $\angle E = \angle E'$	(1) Corr. $\angle$s of $\sim$ $\triangle$s are equal.
(2) $\angle 1 = \angle 1'$ and $\angle 2 = \angle 2'$	(2) Same reason.
(3) $\therefore \angle BCD = \angle B'C'D'$	(3) Ax. 2.
(4) also $\angle CDE = \angle C'D'E'$ and $\angle EAB = \angle E'A'B'$	(4) In same manner as steps (2) and (3).
(5) $\frac{a}{a'} = \frac{b}{b'} = \left(\frac{r}{r'}\right) = \frac{c}{c'} = \left(\frac{s}{s'}\right) = \frac{d}{d'} = \frac{e}{e'}$	(5) Corr. sides of $\sim$ $\triangle$s are in proportion.
(6) $\therefore$ polygon $ABCDE \sim$ polygon $A'B'C'D'E'$	(6) 231.

Q.E.D.

Proposition XVII. Problem

262. Upon a given line corresponding to one side of a given polygon to construct a polygon similar to the given polygon.



Given line $A'B'$ corresponding to side AB of polygon $ABCDE$.

Required to construct on $A'B'$ a polygon $\sim$ polygon $ABCDE$.

Plan is to apply 261.

Construction. (*a*) Draw AC and AD.

(*b*) At B' construct $\angle B'$ equal to $\angle B$, and at A' construct $\angle 1'$ equal to $\angle 1$. We thus get $\triangle A'B'C'$.

(*c*) In like manner construct $\triangle A'C'D'$ and $\triangle A'D'E'$. Then polygon $A'B'C'D'E'$ is the required polygon.

Proof.

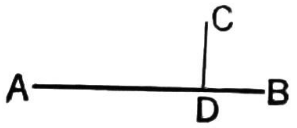
(1) $\triangle ABC \sim \triangle A'B'C'$ $\triangle ACD \sim \triangle A'C'D'$ $\triangle ADE \sim \triangle A'D'E'$	(1) 237.
(2) $\therefore ABCDE \sim A'B'C'D'E'$	(2) 261.

Q.E.D.

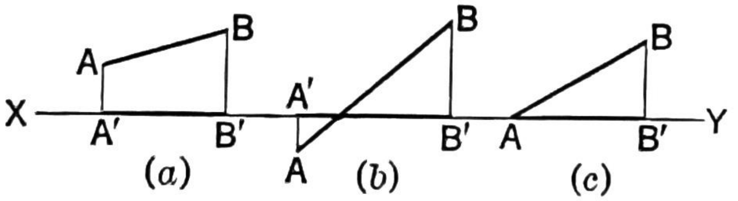
Ex. 527. Given a quadrilateral, required to enlarge it in the ratio 2 to 1 (i.e. construct a $\sim$ quadrilateral whose sides shall be twice as long as those of the given quadrilateral).

Ex. 528. Construct a △ ∼ a given △ and having a given perimeter.

263. *Def.* The *projection* of a point upon a line is the foot of the perpendicular from the point to the line. Thus D is the projection of C upon AB.



264. *Def.* The projection of one line upon another is the segment between the projections of the extremities of the first line upon the second. Thus in (*a*) and (*b*) $A'B'$ is the projection of AB upon XY. In (*c*), AB' is the projection of AB upon XY.



NUMERICAL RELATIONS

Proposition XVIII. Theorem

265. In any triangle the square of the side opposite an acute angle is equal to the sum of the squares of the other two sides diminished by twice the product of one of those sides and the projection of the other side upon it.

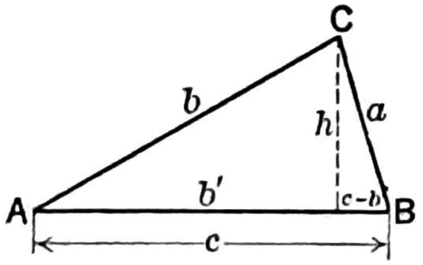


Fig. 1

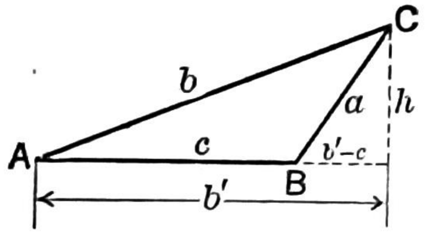


Fig. 2

Given △ ABC with ∠ A acute, and b' denoting the projection of b upon c.

Prove $a^2 = b^2 + c^2 - 2cb'$.

Plan is to use 254.

Proof.

(1) In Fig. 1, $a^2 = h^2 + (c - b')^2$ In Fig. 2, $a^2 = h^2 + (b' - c)^2$	(1) 254.
(2) $\therefore a^2 = h^2 + c^2 - 2\,cb' + b'^2$	(2) Expanding.
(3) but $h^2 + b'^2 = b^2$	(3) 254.
(4) $\therefore a^2 = b^2 + c^2 - 2\,cb'$	(4) Sub. in (2). Q.E.D.

266. Note that the equation $a^2 = b^2 + c^2 - 2\,cb'$ involves four quantities. Hence if any three of them are known, the fourth can be found by solving the equation for that quantity.

Ex. 529. The sides of a triangle are 4″, 5″, 6″. Find the projection of side 6 upon side 4; also the projection of side 4 upon side 6; also the projection of side 4 upon side 5.

Ex. 530. In $\triangle ABC$ given $b = 9''$, $c = 24''$, $\angle A = 60^\circ$. Find a.

Ex. 531. In $\triangle ABC$ given $b = 5''$, $c = 8''$, $\angle A = 45^\circ$. Find a.

Ex. 532. Two sides of a triangle are 15′ and 18′ respectively, and the projection of side 18 upon side 15 is 17″. Find the third side correct to one decimal place.

GROUP XIV

Complete each of the following statements:

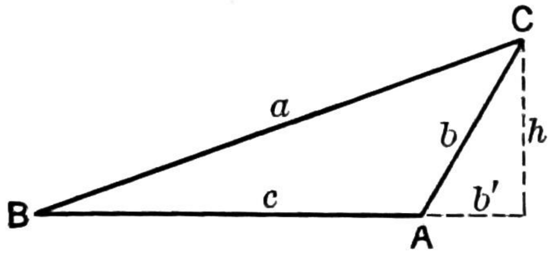
(*a*) In $\triangle ABC$ given $c = 6''$, $\angle A = 120^\circ$, $\angle B = 45^\circ$. To the nearest tenth of an inch h_a is . . . inches, and h_c is . . . inches.

(*b*) The sides of a triangle are 3, 4, $\sqrt{13}$. The angle opposite $\sqrt{13}$ is an angle of . . . degrees.

(*c*) In any 30° right triangle the shorter leg is to the longer leg in the ratio of . . . .

Proposition XIX. Theorem

267. In any obtuse triangle the square of the side opposite the obtuse angle is equal to the sum of the squares of the other two sides increased by twice the product of one of those sides and the projection of the other side upon it.



Given △ *ABC* with ∠ *A* obtuse, and b' denoting the projection of b upon c.

Prove $a^2 = b^2 + c^2 + 2\,cb'$.

Plan is to use 254.

Proof.

(1) $a^2 = h^2 + (c + b')^2$	(1) 254.
(2) $a^2 = h^2 + c^2 + 2\,cb' + b'^2$	(2) Expanding.
(3) but $h^2 + b'^2 = b^2$	(3) 254.
(4) $\therefore\ a^2 = b^2 + c^2 + 2\,cb'$	(4) Sub. in (2). Q.E.D.

268. *Note.* If c is the longest side of a △, ∠ C is the largest angle. Hence if $c^2 < a^2 + b^2$, the △ is acute; if $c^2 = a^2 + b^2$, the △ is right; if $c^2 > a^2 + b^2$, the △ is obtuse. We thus have a method of determining whether a △ is acute, right, or obtuse when the three sides are given.

Exercises

Ex. 533. The sides of a △ are 8″, 10″, 12″. Is the △ acute, right, or obtuse?

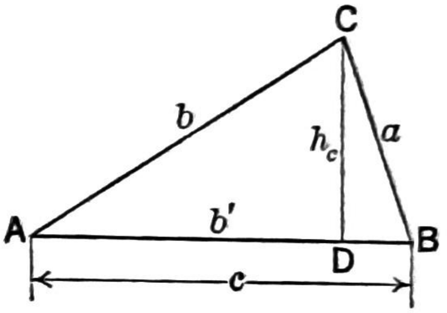
Ex. 534. Determine in each case the kind of triangle when the sides are as follows: 4, 9, 11; 10, 24, 26; 2, 3, 4; 13, 14, 15.

Ex. 535. In $\triangle ABC$ given $b = 3$, $c = 5$, $\angle A = 120°$. Find a.

Ex. 536. Given the three sides of a triangle, derive a formula for the altitude upon one of these sides.

Solution. $a^2 = b^2 + c^2 - 2\,cb'$

$$\therefore b' = \frac{b^2 + c^2 - a^2}{2\,c}$$



$$h_c^2 = b^2 - b'^2$$

$$= b^2 - \left(\frac{b^2 + c^2 - a^2}{2\,c}\right)^2$$

$$= \left(b + \frac{b^2 + c^2 - a^2}{2\,c}\right)\left(b - \frac{b^2 + c^2 - a^2}{2\,c}\right)$$

$$= \frac{2\,bc + b^2 + c^2 - a^2}{2\,c} \cdot \frac{2\,bc - b^2 - c^2 + a^2}{2\,c}$$

$$= \frac{(b + c)^2 - a^2}{2\,c} \cdot \frac{a^2 - (b - c)^2}{2\,c}$$

$$= \frac{(b + c + a)(b + c - a)(a + b - c)(a - b + c)}{4\,c^2}$$

$$= \frac{2\,s(2\,s - 2\,a)(2\,s - 2\,c)(2\,s - 2\,b)}{4\,c^2} \quad \text{where } a + b + c = 2\,s$$

$$= \frac{4\,s(s - a)(s - b)(s - c)}{c^2}$$

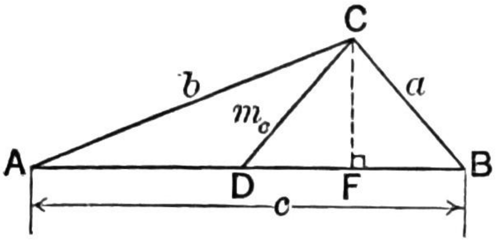
$$\therefore h_c = \frac{2}{c}\sqrt{s(s - a)(s - b)(s - c)}$$

Ex. 537. The sides of a $\triangle$ are 6″, 9″, 12″. Find correct to one decimal place the altitude upon side 6.

Sug. Use the formula derived in Ex. 536.

Ex. 538. Given the three sides of a triangle, derive a formula for the median to one of these sides.

Solution.



$$a^2 = m_c{}^2 + \left(\frac{c}{2}\right)^2 - 2\left(\frac{c}{2}\right)\cdot DF$$

$$b^2 = m_c{}^2 + \left(\frac{c}{2}\right)^2 + 2\left(\frac{c}{2}\right)\cdot DF$$

$$\therefore\ a^2 + b^2 = 2\,m_c{}^2 + 2\left(\frac{c}{2}\right)^2$$

$$2\,a^2 + 2\,b^2 = 4\,m_c{}^2 + c^2$$

$$4\,m_c{}^2 = 2\,a^2 + 2\,b^2 - c^2$$

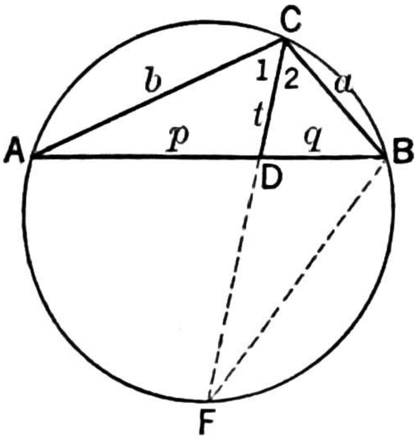
$$\therefore\ m_c = \tfrac{1}{2}\sqrt{2\,a^2 + 2\,b^2 - c^2}$$

Ex. 539. The sides of a △ are 9″, 10″, 11″. Find the median to side 10.

Ex. 540. Prove that the sum of the squares of the four sides of a parallelogram is equal to the sum of the squares of the diagonals.

Proposition XX. Theorem

269. In any triangle the product of two sides is equal to the square of the bisector of the included angle increased by the product of the segments of the third side.



Given △ *ABC*, *t* the bisector of ∠ *C* and forming segments *p* and *q* on *AB*.

Prove $ab = t^2 + pq$.

Analysis. pq suggests the product of the segments of a chord. Hence circumscribe a ⊙ about △ ABC, and prolong CD to meet the ⊙ at F. ab suggests that we get two ∼ ⚠ of which a and b are respective sides. A study of the figure indicates that △ ACD and △ CBF are similar.

Proof. Circumscribe a ⊙ about △ ABC, prolong CD to meet this ⊙ at F. Draw BF. Then

(1) $\angle 1 = \angle 2$	(1) Given.
(2) $\angle A = \angle F$	(2) Each is measured by $\frac{1}{2}\,\widehat{BC}$, by 176.
(3) $\therefore \triangle ACD \sim \triangle CBF$	(3) 237.
(4) $\therefore b : t = t + DF : a$	(4) Corr. sides of ∼ ⚠ are in proportion.
(5) $\therefore ab = t^2 + t \cdot DF$	(5) 213.
(6) but $t \cdot DF = pq$	(6) 247.
(7) $\therefore ab = t^2 + pq$	(7) Sub.

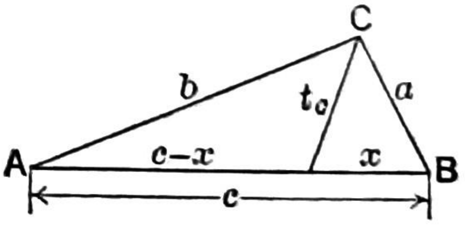
Q.E.D.

Ex. 541. The sides of a △ are 3″, 6″, 7″. Find the length of the bisector of the angle opposite side 7.

Sug. To obtain the segments on side 7 use 230.

Ex. 542. Given the three sides of a triangle, derive a formula for the length of the bisector of any angle of the triangle.

Solution.



$$ab = t_c^2 + x(c - x)$$

$$\therefore t_c^2 = ab - x(c - x)$$

Now $$\frac{x}{c - x} = \frac{a}{b}$$

$$\therefore x = \frac{ac}{a + b}$$

$$\therefore c - x = c - \frac{ac}{a + b} = \frac{bc}{a + b}$$

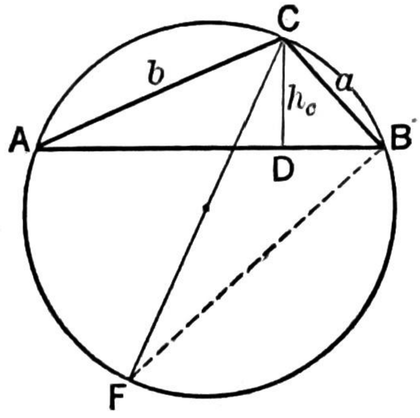
$$\therefore t_c^2 = ab - \frac{ac}{a+b} \cdot \frac{bc}{a+b}$$
$$= ab - \frac{abc^2}{(a+b)^2}$$

This result may be transformed into a form more suitable for computation, as follows:

$$t_c^2 = ab\left[1 - \frac{c^2}{(a+b)^2}\right]$$
$$= ab\left(1 + \frac{c}{a+b}\right)\left(1 - \frac{c}{a+b}\right)$$
$$= ab\left(\frac{a+b+c}{a+b}\right)\left(\frac{a+b-c}{a+b}\right)$$
$$= \frac{ab}{(a+b)^2} \cdot 2\,s \cdot (2\,s - 2\,c) \text{ where } a + b + c = 2\,s$$
$$= \frac{4}{(a+b)^2} \cdot abs(s-c)$$
$$\therefore t_c = \frac{2}{a+b}\sqrt{abs(s-c)}$$

Proposition XXI. Theorem

270. In any triangle the product of two sides is equal to the product of the diameter of the circumscribed circle and the altitude upon the third side.



Given $\triangle ABC$, d a diameter of the $\odot$ circumscribed about $\triangle ABC$, and h_c the altitude upon c.

Prove $ab = dh_c$.

Analysis. The required equation suggests a proportion, and a proportion suggests $\sim$ $\triangle$s *involving the four quantities.*

Proof. Draw BF. Then

(1) $\angle ADC$ is a right angle.	(1) Given.
(2) $\angle CBF$ is a right angle.	(2) 177.
(3) $\angle A = \angle F$	(3) Each is measured by $\frac{1}{2}\overset{\frown}{BC}$, by 176.
(4) $\therefore \triangle ADC \sim \triangle CBF$	(4) 238.
(5) $\therefore b : h_c = d : a$	(5) Corr. sides of $\sim$ $\triangle$s are in proportion.
(6) $\therefore ab = dh_c$	(6) 213.

Q.E.D.

Ex. 543. Compute Ex. 541 by using the formula derived in Ex. 542.

Ex. 544. Given the three sides of a $\triangle$, derive a formula for the diameter of the circumscribed $\odot$.

Solution.

$$ab = d \cdot h_c = d \cdot \frac{2}{c}\sqrt{s(s-a)(s-b)(s-c)}$$

$$\therefore abc = 2\,d\sqrt{s(s-a)(s-b)(s-c)}$$

$$\therefore d = \frac{abc}{2\sqrt{s(s-a)(s-b)(s-c)}}$$

Ex. 545. The sides of a $\triangle$ are 6″, 7″, 11″. Find the radius of the circumscribed $\odot$.

Miscellaneous Exercises — Theorems

Ex. 546. If two Ⓢ are tangent to each other, the chords formed by a straight line drawn through the point of contact are to each other as the diameters of the Ⓢ.

Ex. 547. AB is a diameter of a $\odot$ and F is any point in the semicircle *AFB*. If *AF* and *BF* produced meet the tangents at *B* and *A* in points *C* and *D* respectively, prove that *AB* is the mean proportional between *AD* and *BC*.

Ex. 548. If one acute angle of a right triangle is double the other acute angle, prove that the square of the longer leg equals three times the square of the shorter leg.

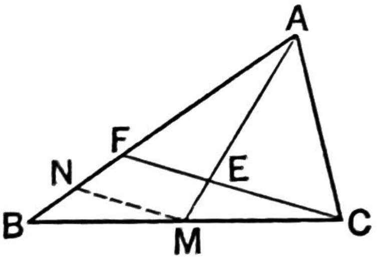
Ex. 549. The bisectors of corresponding angles of two similar triangles are to each other as any two corresponding medians.

Ex. 550. The midpoint of an arc subtended by a chord is joined to the extremities of another chord. Prove that the triangles thus formed are similar.

Ex. 551. Two tangents are drawn from a point to a ⊙. Prove that the square of the radius of the ⊙ is equal to the product of the line joining the external point to the center multiplied by the inner segment of that line made by the chord of contact.

Ex. 552. In rectangle *ABCD* a perpendicular is drawn from *D* upon *AC*, the prolongation of which intersects *AB* in *E*. Prove $\overline{AD}^2 = AE \times CD$.

Ex. 553. *M* is the midpoint of side *BC* of △ *ABC*, and any line is drawn from *C* cutting *AM* in *E* and *AB* in *F*. Prove $\frac{AE}{EM} = 2\left(\frac{AF}{FB}\right)$.



Sug. Draw *MN* ∥ *CF*.

Ex. 554. *AC* and *BC* are legs of the isosceles △ *ABC*. *AF* is the altitude on *BC*. Prove $\overline{AB}^2 = 2\,BC \times BF$.

Ex. 555. The difference of the squares of two sides of a triangle is equal to the difference of the squares of the segments of the third side made by the altitude upon that side.

Ex. 556. If the altitudes of △ *ABC* meet at *O*, prove that $\overline{AB}^2 - \overline{AC}^2 = \overline{OB}^2 - \overline{OC}^2$.

Ex. 557. *M* is the midpoint of side *BC* of △ *ABC*, and the bisectors of angles AMB and AMC meet AB and AC in E and F respectively. Prove $EF \parallel BC$.

Ex. 558. Two ⓢ are tangent at P, and through P three lines are drawn cutting the ⓢ in A, B, C and A', B', C' respectively. Prove $\triangle ABC \sim \triangle A'B'C'$.

Ex. 559. If two sides and the radius of the circumscribed ⊙ of one triangle are proportional to the corresponding parts of another triangle, prove that the two triangles are similar.

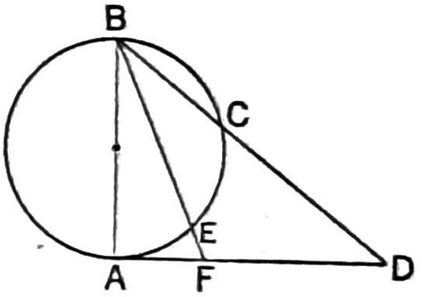
Ex. 560. If the square of one side of a triangle is equal to the sum of the squares of the other two sides, the triangle is right.

Sug. Draw a right △ whose legs are respectively equal to the legs of the given △, and prove the two ⚠ congruent.

Ex. 561. If the square of one side of a triangle is greater than the sum of the squares of the other two sides, the triangle is obtuse.

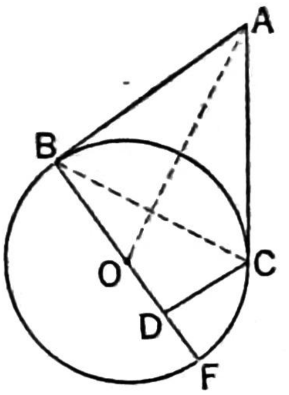
Sug. Apply 120.

Ex. 562. In the annexed drawing AB is a diameter, AD a tangent, FB and DB are secants. Prove $BE \times BF = BC \times BD$.



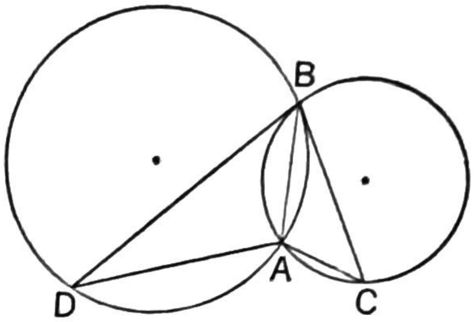
Ex. 563. AB and AC are tangents to ⊙ O from point A. $CD \perp$ diameter BF. Prove $AB : OB = BD : CD$.

Sug. Prove $AO \perp BC$.



Ex. 564. The sum of the squares of the distances of any point on a circle from the vertices of an inscribed square is equal to twice the square of the diameter of the circle.

Ex. 565. Two Ⓢ intersect in A and B, and the two chords BD and BC are respectively tangents to the two Ⓢ. Prove AB is the mean proportional between AD and AC.



Ex. 566. The radii of the inscribed circles of two similar triangles are to each other as any two corresponding sides.

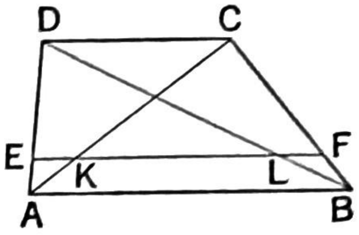
Ex. 567. The square of the line joining the center of a circle to any point of a chord plus the product of the segments of the chord is equal to the square of the radius.

Ex. 568. M is the midpoint of BC, one of the parallel sides of the trapezoid $ABCD$. AM and DM produced meet DC and AB produced at F and E respectively. Prove $EF \parallel AD$.

Ex. 569. If two circles are tangent externally, their common external tangent is the mean proportional between the diameters.

Ex. 570. $ABCD$ is a trapezoid, and EF is any line parallel to the bases and cutting AC in K and BD in L. Prove $EK = LF$.

Sug. Prove $EK : DC = LF : DC$.



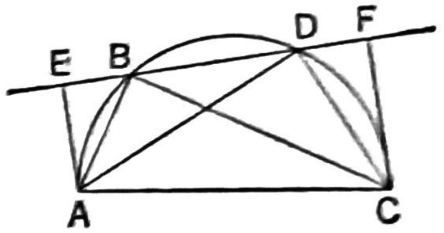
Ex. 571. If two tangents are drawn to a circle at the extremities of a diameter, the portion of a third tangent intercepted between them is divided at its point of contact into segments whose product is equal to the square of the radius.

Sug. Apply 252 II.

Ex. 572. AB is the hypotenuse of right $\triangle ABC$. At A and B perpendiculars to AB are drawn meeting BC produced in D and AC produced in F. Prove $CA : CF = CD : CB$.

Ex. 573. If AB and CD intersect in F so that $AF \times FB = CF \times FD$, prove that A, C, B, D lie in a ⊙.

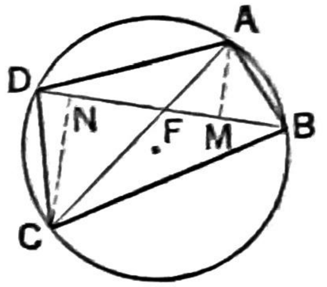
Sug. Prove that the ⊙ which passes through A, C, B cannot pass above D nor below D. ∴ it must pass through D.



Ex. 574. ABC and ADC are angles inscribed in a semicircle. AE and CF are ⊥ BD. Prove $\overline{BE}^2 + \overline{BF}^2 = \overline{DE}^2 + \overline{DF}^2$.

Ex. 575. Two chords AB and CD intersect in M, A is the midpoint of the arc CD. Prove that $AB \times AM$ is constant if the chord AB is made to turn about the fixed point A.

Sug. Prove $AB \times AM$ equals the product of two fixed distances. To do this, draw diameter AE and also draw BE.



Ex. 576. $ABCD$ is a quadrilateral inscribed in a ⊙. The diagonals AC and BD intersect in F. Prove $\frac{AB \times AD}{CB \times CD} = \frac{AF}{FC}$.

Sug. Draw $AM \perp DB$, $CN \perp DB$, and then apply 270.

Ex. 577. The squares of two chords drawn from the same point on a circle have the same ratio as the projections of the chords on the diameter drawn from the same point.

Sug. Apply 252 III.

Ex. 578. $ABCD$ is a trapezoid. The non-parallel sides AD and BC are produced to meet at F. The diagonals AC and BD meet at O. FO prolonged meets AB at H. Prove $AH = HB$.

Sug. Prove $AH : HB = HB : AH$.

Ex. 579. If in a right triangle the medians upon the legs are drawn, prove that five times the square of the hypotenuse is equal to four times the sum of the squares of the two medians.

Ex. 580. The non-parallel sides of a trapezoid and the line joining the midpoints of the parallel sides, if produced, meet in one point.

Ex. 581. The square of a leg of an isosceles triangle is equal to the square of any line drawn from the vertex to the base increased by the product of the segments of the base.

Ex. 582. If from one of the acute angles of a right triangle a line is drawn bisecting the opposite side, prove that the square of this line is less than the square of the hypotenuse by three times the square of half the side that is bisected.

Ex. 583. If lines are drawn from any point P to the vertices of rectangle $ABCD$, prove that $\overline{PA}^2 + \overline{PC}^2 = \overline{PB}^2 + \overline{PD}^2$.

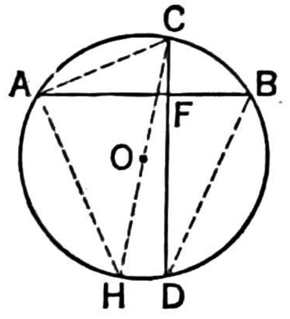
Ex. 584. AB and AC are legs of an isosceles $\triangle ABC$. The perpendicular bisector of AB meets BC in D. Prove $\overline{AB}^2 = BC \times BD$.

Ex. 585. The sum of the squares of the four sides of any quadrilateral is equal to the sum of the squares of the diagonals plus four times the square of the line joining the midpoints of the diagonals.

Sug. Apply Ex. 538.

Ex. 586. The sum of the squares of the segments of two perpendicular chords is equal to the square of the diameter of the circle.

Sug. Prove $BD = AH$.



Ex. 587. AB and AC are divided at points D and E

respectively so that $AB \times AD = AC \times AE$. Prove that B, C, E, D lie in a ⊙ (are concyclic).

Sug. By use of ∼ ▵ prove $\angle DBE = \angle ECD$, and then apply Ex. 306.

Ex. 588. If the medians upon two sides of a triangle are equal, prove the triangle is isosceles.

Numerical Exercises

Ex. 589. Find the side of a square whose diagonal is 8″.

Ex. 590. A tower casts a shadow 125′ long when a vertical 15′ pole casts a shadow 12′ long. How high is the tower?

Ex. 591. The legs of a right triangle are in the ratio 5 : 4, and the hypotenuse is 30′. Find the legs.

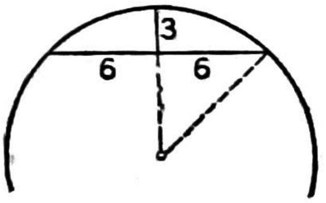
Ex. 592. A walks 6 miles north, then 5 miles east, and then 3 miles north. How far is he from the starting point?

Ex. 593. The bases of a trapezoid are 7″ and 11″, the legs are 3″ and 6″ respectively. How far must each of the legs be produced so that they meet at a point?

Ex. 594. Compute the altitude of the above trapezoid.

Ex. 595. The legs of a right triangle are in the ratio 3 : 1. What is the ratio of the segments of the hypotenuse formed by the altitude upon the hypotenuse?

Ex. 596. The diameter of a ⊙ is 10′. A chord of this ⊙ is 6′ long and is perpendicular to the diameter. How far is the chord from the center of the ⊙?



Ex. 597. The height of a circular arch is 3′ and its span is 12′. What is the radius?

Ex. 598. Chords drawn from a point on a ⊙ to the extremities of a diameter are respectively 10″ and 12″. Find the distance of the point from the diameter.

Ex. 599. The bases of a trapezoid are 8″ and 12″, the altitude is 5″. Find the length of a line drawn between the legs parallel to the lower base and 2″ from it.

Ex. 600. The angles of a triangle are 30°, 60°, 90°. Find the ratio of the segments into which the side opposite the 60° angle is divided by the bisector of that angle.

Ex. 601. AB is a line $25''$ long. $AD \perp AB$, $BE \perp AB$, $AD = 13''$, $BE = 7''$. In AB produced a point C is taken so that $\angle DCA = \angle EAB$. Find CA and CB.

Ex. 602. Find the legs of a right triangle if their projections on the hypotenuse are $3'$ and $6'$ respectively.

Ex. 603. How far from the center of a ⊙ must a point be taken so that the tangents to the ⊙ from that point shall form an angle of 60°? of 90°? of 120°?

Ex. 604. Each of the legs of an isosceles triangle is $6'$. The angle included between them is 30°. Find the length of the base. How long is the base when the angle included between the legs is 45°?

Ex. 605. The radii of two Ⓢ are $9''$ and $5''$ respectively, and the distance between their centers is $20''$. How far from the center of the smaller ⊙ does the common external tangent cut the line of centers? the common internal tangent?

Ex. 606. The base of a triangle is $18''$, the other sides are $9''$ and $16''$. A line parallel to the base and terminating in the sides cuts off $3''$ from the lower end of the shorter side. Find the segments of the other side and the length of the parallel.

Ex. 607. Two sides of a triangle are $8''$ and $12''$, and the altitude on the third side is $6''$. A similar triangle has the side corresponding to 8 equal to $10''$. Compute the other two sides and the altitude to the longest side in the second triangle.

Ex. 608. Find the radius of the ⊙ inscribed in an equilateral triangle whose side is $10''$.

Ex. 609. Two sides of a triangle are $7''$ and $12''$. The median on side 12 is $4\frac{1}{2}''$. Find the third side.

Ex. 610. The base of a triangle is $38''$ and the altitude upon it is 15″. A line is drawn parallel to the base, terminating in the other two sides and 11″ from it. Find the length of this line.

Ex. 611. From a point 13″ from the center of a ⊙ whose radius is 12″ a secant is drawn which is 5″ distant from the center. Find the length of the external segment of the secant.

Ex. 612. The diagonals of a parallelogram are 26″ and 28″, the base is 12″. Find the altitude.

Ex. 613. The bases of an isosceles trapezoid circumscribed about a ⊙ are 18″ and 6″. Find the diameter of the ⊙.

Ex. 614. The angle between two tangents to a ⊙ is 60°, and the chord of contact is 8″. Find the radius of the ⊙.

Ex. 615. In an isosceles triangle the base is 13″ and each leg is 11″. Find the radius of the circumscribed ⊙.

Ex. 616. In $\triangle ABC$, $AB = 12''$; $AC = 8''$, $BC = 8''$. D is a point on AC such that $AD = 3''$. Find the length of BD.

Ex. 617. The legs of an inscribed right triangle are 6″ and 8″. Find the length of the chord that bisects the right angle.

Ex. 618. The sides of a triangle are 3″, 7″, 8″. Find the angle opposite side 7.

Ex. 619. The sides of a triangle are 7″, 9″, 15″.

(*a*) Determine whether the triangle is right, acute, or obtuse.

(*b*) Find the altitude on side 7.

(*c*) Find the median to side 9.

(*d*) Find the bisector of the largest angle.

(*e*) Find the radius of the circumscribed circle.

Miscellaneous Construction Problems

Ex. 620. Divide a given line into two parts which shall be to each other as $1 : \sqrt{2}$.

Sug. The side of a square is to its diagonal as $1 : \sqrt{2}$.

Ex. 621. Construct a line that shall be to a given line (a) as $1 : \sqrt{2}$, (b) as $1 : \sqrt{3}$, (c) as $1 : \sqrt{5}$.

Ex. 622. Construct a triangle having given a, $b + c$, and $b : c = 3 : 2$.

Sug. $\therefore b = \frac{3}{5}(b + c)$, $c = \frac{2}{5}(b + c)$.

Ex. 623. Construct a triangle having given a, b, and the ratio $a : c$.

(See Sug. to Ex. 434.)

Ex. 624. In the $\triangle ABC$ draw a line parallel to AB cutting AC in D and BC in E so that $AB : DE = m : n$.

Ex. 625. Locate a point within a given angle such that its distance from the sides of the angle shall be in the ratio $m : n$.

Ex. 626. In a given ⊙ inscribe a triangle similar to a given triangle.

Sug. Circumscribe a ⊙ about the given $\triangle$.

Ex. 627. About a given ⊙ circumscribe a triangle similar to a given triangle.

Sug. Inscribe a ⊙ in the given $\triangle$.

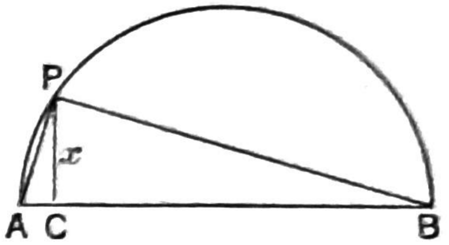
Ex. 628. Construct a triangle similar to a given triangle and having a given altitude.

Ex. 629. D is a given point in the side AB of $\triangle ABC$. Required to find a point X in the prolongation of AC such that DX is bisected by BC.

Ex. 630. Construct a triangle, given b and the segments of c made by the bisector of $\angle C$.

Ex. 631. Divide a semicircle into two arcs such that the radius shall be a mean proportional between the chords of the arcs.

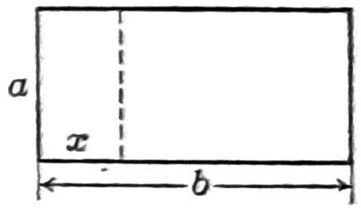
Analysis. Assume the problem done, and let P be the required point. Draw $PC \perp AB$, and denote it by x. Then $PA \cdot PB = r^2$; but $PA \cdot PB = AB \cdot x = 2\,rx$; $\therefore 2\,rx = r^2$; $\therefore x = \frac{1}{2}\,r$; $\therefore$ etc.



Ex. 632. Through one of the points of intersection of two circles draw a secant such that the two chords thus formed shall be to each other as $m : n$.

Sug. Divide the centers-segment in the ratio $m : n$, etc.

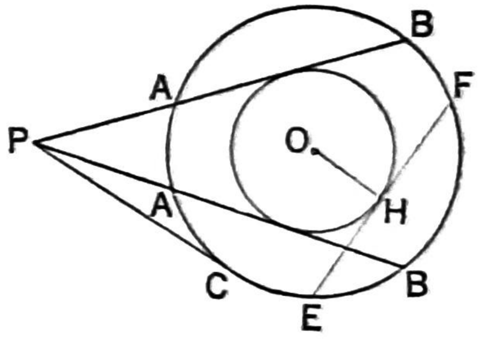
Ex. 633. From a given rectangle cut off a similar rectangle by a line parallel to one side.



Sug. If we assume the problem done, then $b : a = a : x$.

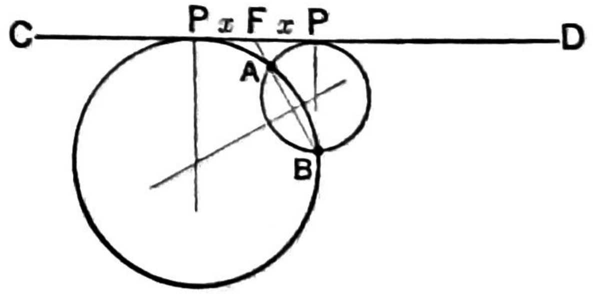
Ex. 634. Through a given external point P construct a secant PAB to a given circle so that $\overline{AB}^2 = PA \cdot PB$.

Analysis. Assume the problem done. Then $PA \cdot PB = \overline{PC}^2$; but $PA \cdot PB = \overline{AB}^2$; $\therefore PC = AB$. Hence draw EF any chord of $\odot\, O$ such that its length equals the tangent PC. Draw $OH \perp EF$, and with O as center and OH as radius draw a $\odot$. Draw PB tangent to this $\odot$. Then PAB is the required secant. When is there one solution? When no solution?



Ex. 635. Construct a circle which shall be tangent to a given line and pass through two given points.

Analysis. CD is the given line; A and B the given points. Assume the problem done. Then $FB : x = x : FA$. This locates the point of contact P. There are two solutions.



Ex. 636. Construct a circle which shall be tangent to two given parallel lines and pass through a given point.

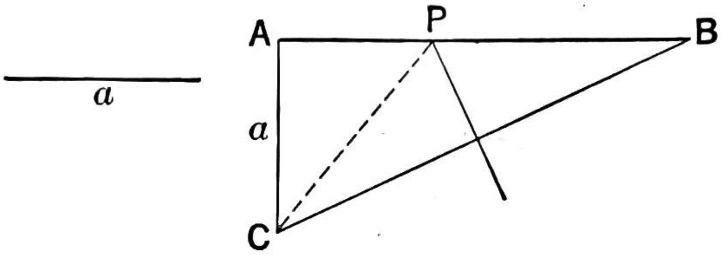
Ex. 637. Locate a point P in the arc AB of $\odot O$ such that chord PA : chord $PB = m : n$.

Sug. Divide chord AB in the ratio $m : n$. Then apply 230.

Ex. 638. Divide a given line into two segments such that the difference of their squares shall equal a^2.

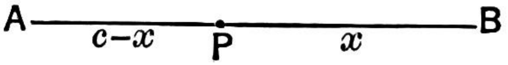
Solution. AB and a are given lines.

Draw $AC \perp AB$, and make its length equal to a. Draw CB and then let the $\perp$ bisector of CB meet AB at P. Then P is the required point.



Proof. $\overline{PB}^2 - \overline{PA}^2 = \overline{PC}^2 - \overline{PA}^2 = a^2$.

Note: This problem can also be solved by algebraic analysis, as follows: Let the length of AB be denoted by c. Assume the problem done, let P be the required point, denote PB by x, and AP by $c - x$. Then



$$x^2 - (c - x)^2 = a^2$$
$$x^2 - c^2 + 2\,cx - x^2 = a^2$$
$$\therefore x = \frac{a^2 + c^2}{2\,c}$$
$$\therefore cx = \tfrac{1}{2}\sqrt{a^2 + c^2} \cdot \sqrt{a^2 + c^2}$$
$$\therefore c : \sqrt{a^2 + c^2} = \tfrac{1}{2}\sqrt{a^2 + c^2} : x$$

$\therefore x$ is the fourth proportional between the first three quantities. As a matter of fact, one way of constructing this fourth proportional is by the solution given above.

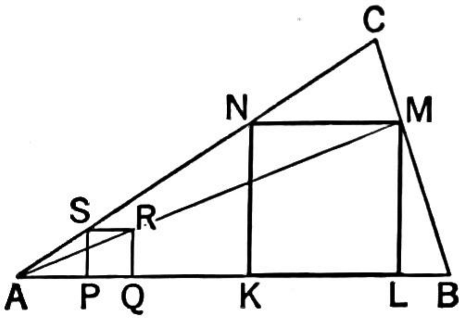
Ex. 639. Divide a given line into two parts so that the square of one part shall equal half the square of the other.

Sug. This amounts to dividing the given line in the ratio $1 : \sqrt{2}$.

Ex. 640. From a given point outside a circle draw a secant such that its external segment shall equal the internal segment.

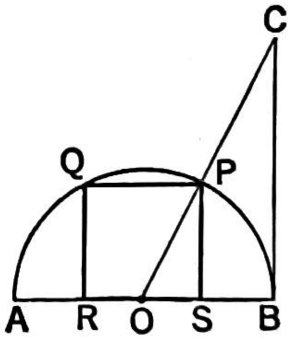
Ex. 641. Inscribe a square in a given triangle.

Solution. At P, any point in AB, draw a perpendicular to AB and let it meet AC at S. On PS as a side erect a square $PQRS$. Draw AR and prolong it till it meets BC at M. This locates one point of the required square. Draw $MN \parallel AB$, draw ML and $NK \perp AB$. Then $KLMN$ is the required square. The proof is based on similar △. Let the pupil give it in full.



Ex. 642. Inscribe a square in a given semicircle.

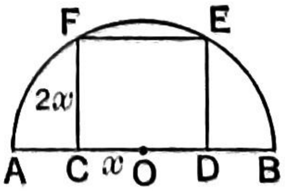
Solution. Draw $BC \perp AB$ and equal to AB. Draw OC and note that it cuts the semicircle at P. Draw $PQ \parallel AB$, and draw PS and $QR \perp AB$. Then $PQRS$ is the required square. Let the pupil give the complete proof. This problem could also readily be solved by algebraic analysis. Thus: Assume the problem done, and let $OC = x$, $OA = r$. Then



$$r - x : 2x = 2x : r + x$$

$$\therefore 4x^2 = r^2 - x^2$$

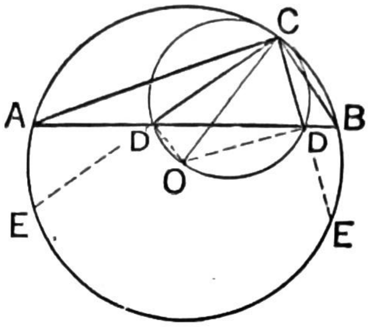
$$x^2 = \frac{r^2}{5}.$$



$\therefore r : x = x : \frac{1}{5} r.$ Hence OC can be constructed, etc.

Ex. 643. Draw a line from the vertex of the obtuse angle of an obtuse triangle to the opposite side so that it shall be the mean proportional between the segments of that side.

Solution. Circumscribe a ⊙ about △ *ABC*. On *OC* as diameter draw a ⊙. This cuts *AB* at *D* and *D*. Then *CD* is the required line, and we have two solutions. For proof, apply 247.



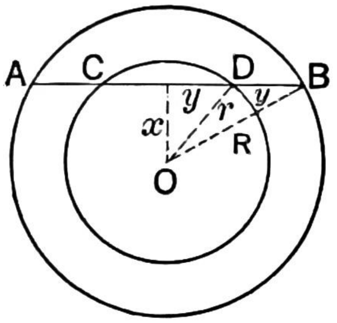
Ex. 644. Draw a line cutting two concentric Ⓢ so that the chord intercepted by the greater ⊙ shall be twice the chord intercepted by the smaller ⊙.

Analysis. Assume the problem done, and let *AB* be the required chord. Then

$$\begin{cases} x^2 + y^2 = r^2 \\ x^2 + 4y^2 = R^2 \end{cases}$$

$$\therefore 3x^2 = 4r^2 - R^2$$

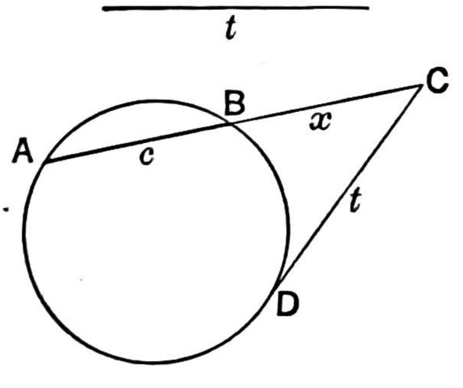
$$\therefore \tfrac{1}{3}(2r + R) : x = x : 2r - R.$$

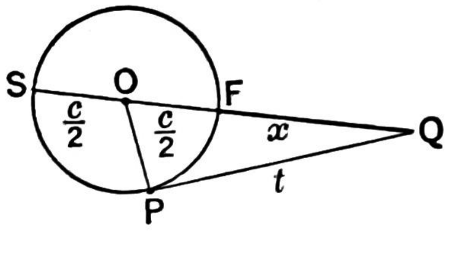


Hence x can be constructed. The problem has an infinite number of solutions.

Ex. 645. Produce a given chord of a ⊙ so that the tangent drawn from its extremity shall have a given length.

Analysis. Assume the problem done. Then $x(c + x) = t^2$. x can be constructed geometrically as follows: Draw a ⊙ with diameter c. At any point P on the ⊙ draw a tangent of length t. Draw *QOS*. Then *QF* is the length x.





Ex. 646. Find the locus of a point such that the tangents drawn from it to two given intersecting circles shall be equal.

Ex. 647. Find the locus of a point whose distances from two given intersecting lines are to each other as $m : n$.

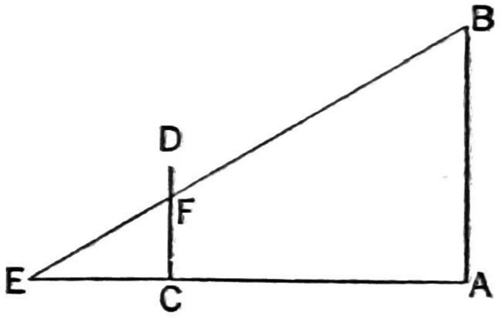
Hint. The locus consists of two straight lines through the intersection of the two given lines.

Practical Applications

Ex. 648. The sides of a triangular field are found by actual measurement to be 120 rds., 168 rds., and 210 rds. Make a drawing of the field to the scale 1″ = 50 rds.

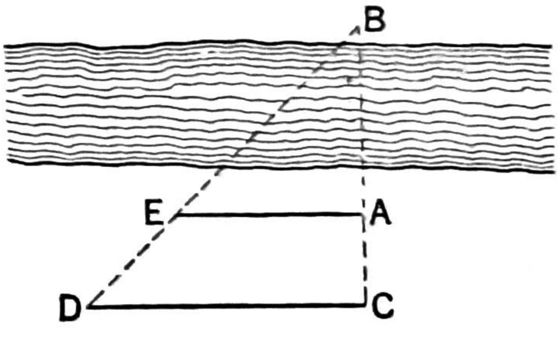
Ex. 649. The shadow of a tree is 40′ long at the same time that the shadow of a 4′ pole is 2′ 3″. What is the height of the tree?

Ex. 650. How may the height of a flagpole be found by noticing just when the length of the shadow of a certain post is equal to the height of the post?

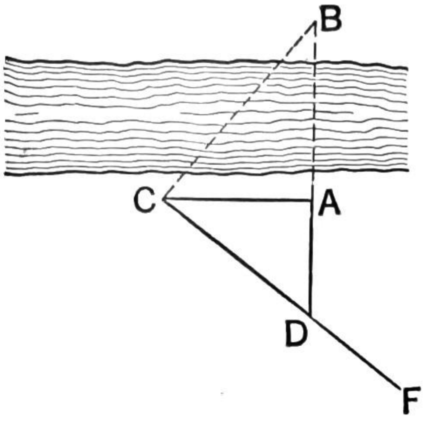


Ex. 651. To obtain the height of a tower AB, a boy measured the distance EA from the foot of the tower, then measured off a convenient distance EC in line with EA; at C another boy held a stick CD vertically and marked on it the point F where E, F, and B are in line (the first boy is sighting from E to B). How can they calculate the height AB?

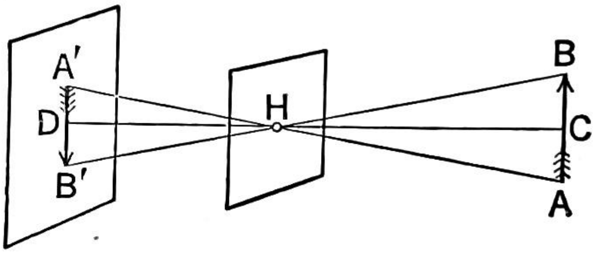
Ex. 652. To find the distance AB across a river, proceed as follows: Measure AE at right angles to AB. Select a point C in line with AB. Measure AC, and then turn a right angle at C. In CD select D in line with E and B. Measure CD. If AE = 40 yds., AC = 15 yds., CD = 50 yds., find AB.



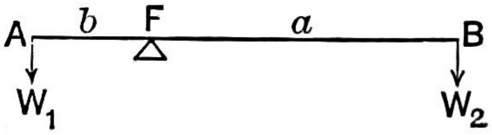
Ex. 653. To find the distance AB across a river, AC is measured at right angles to AB, then CF is turned at right angles to CB, and the point D in CF is located such that D, A, B are in line. AC is 30 yds., and AD is 20 yds. How much is AB?



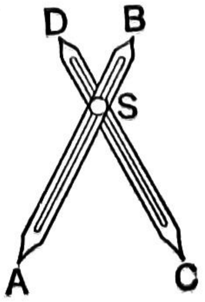
Ex. 654. If light proceeding from any luminous object be allowed to pass through a minute hole in an opaque screen, and fall upon a white sheet in a dark room, an inverted picture of the object will be formed. If $CH = 8'$, $HD = 6'$, $AB = 5'$, find $A'B'$.

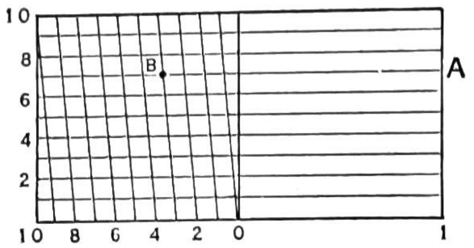


Ex. 655. The lever AB is in balance if $W_1 : W_2 = a : b$. (W_1 and W_2 are downward forces acting at A and B. F is the turning point, or fulcrum. The distances a and b are called the arms of the lever.) A lever is 2 ft. long. Where must the fulcrum be placed so that 10 pounds at one end may balance 30 pounds at the other end?



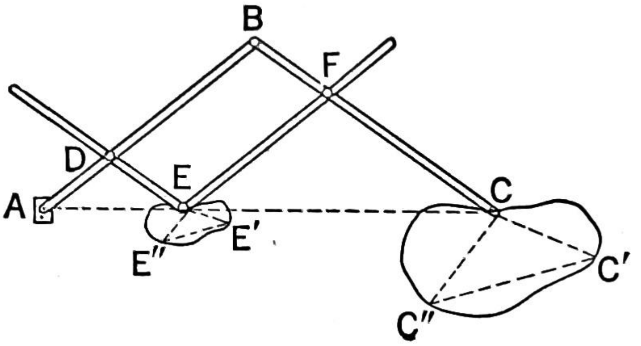
Ex. 656. The adjoining figure shows a pair of proportional dividers. It is made of two equal graduated bars AB and CD having slots such that the adjusting screw S can be placed in the desired position and such that $SB = SD$. If $AS : SB = 3 : 1$, why is $AC : DB = 3 : 1$? Hence if we wish to enlarge a drawing in the ratio 3 : 1, how would we use the dividers?





Ex. 657. The adjoining diagonal scale can be used to make measurements to the hundredth of an inch. Why is AB 1.37″? Show how to measure 1.15″, 1.28″, 1.79″, 1.05″, 1.47″, 1.66″.

Ex. 658. A pantograph is an instrument for drawing a plane figure similar to a given figure, and is therefore useful for enlarging or reducing maps and drawings. It consists of four bars, parallel in pairs, and jointed at D, B, E, and F. A is fixed, and the bars are adjustable at D and F so that $\frac{AD}{AB} = \frac{BF}{BC}$ and $DBFE$ is always a parallelogram. While a pencil at E traces a certain figure, a pencil at C will trace a similar figure such that corresponding lines of these figures are in the fixed ratio $AD : AB$.



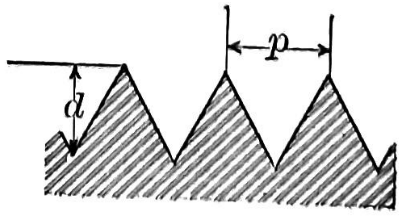
(*a*) Why are A, E, C always in a straight line?

(*b*) If when one pencil is at E' or E'' the other pencil is at C' or C'', why are EE', $E'E''$, $E''E$ parallel respectively to CC', $C'C''$, $C''C$?

(*c*) Why is it true that $\frac{EE'}{CC'} = \frac{E'E''}{C'C''} = \frac{E''E}{C''C} = \frac{AD}{AB}$?

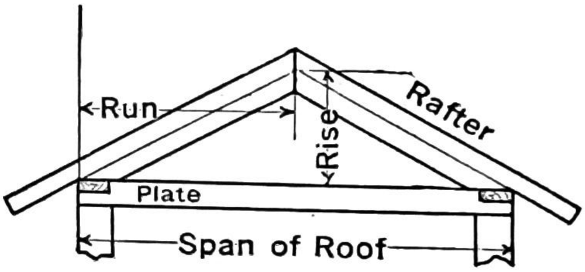
(*d*) Why is $\triangle CC'C'' \sim \triangle EE'E''$?

Ex. 659. The cross-section of a V-thread on a screw is an equilateral triangle. The distance from the top of one thread to the top of the next is called the pitch, and



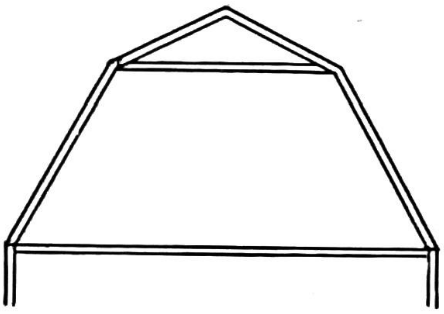
the distance from the top to the bottom of the thread is called the depth. Find the depth of a V-thread of $\frac{1}{8}''$ pitch. Express your result to the nearest hundredth of an inch.

Ex. 660. What length of rafters is required, and what is the pitch of a common gable roof, if the width of the building is 24′ and the rise of the roof is 8′, allowing the rafters to project 18″ over the side of the building at the eaves?

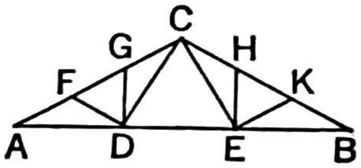


Note: The pitch of a roof means its slant or slope. It is equal to the rise divided by the span.

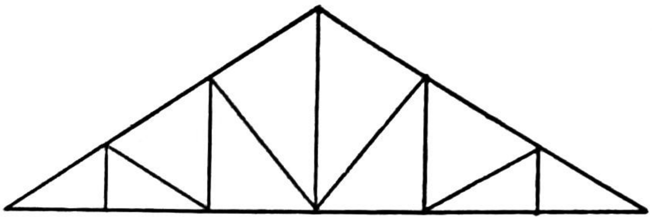
Ex. 661. A gambrel-roof house is 24′ wide, the bottom set of rafters has $\frac{2}{3}$ pitch, and the top set $\frac{1}{3}$ pitch. If the break or joint in the roof comes 7′ in from the side of the building, how long is each set of rafters?



Ex. 662. The triangular roof truss here shown is 60′ long, and CDE is an equilateral triangle. The bottom member and the rafters are divided into equal segments. Find the length of each member.

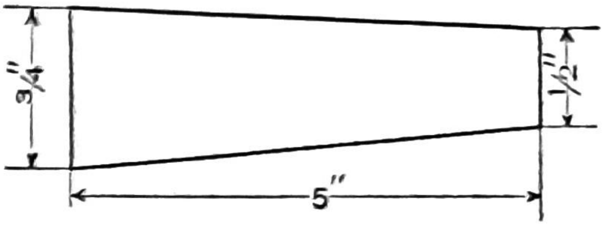


Ex. 663. The roof truss here shown is 90′ long and 30′ high. The bottom member is divided into six equal segments by vertical members. Find the length of the various parts.

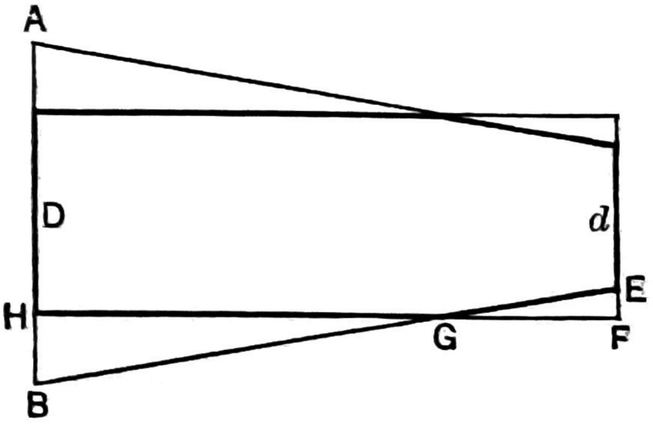


Ex. 664. A piece of work which decreases gradually in

diameter so that it assumes a conical shape is said to be tapered. The taper on a piece of work is the difference between the diameters at the ends of the piece, and is usually designated as so much per foot of length. The length is measured along the axis or center line of the piece of work. Thus in the annexed figure the taper = $\frac{3}{4}'' - \frac{1}{2}'' = \frac{1}{4}''$. $\therefore$ the taper per inch $= \frac{1}{4} \div 5 = \frac{1}{20}''$. $\therefore$ the taper per foot $= 12 \times \frac{1}{20} = \frac{3}{5}''$. Find the large diameter of a piece of tapered work $4\frac{1}{8}''$ long, having a taper of $\frac{5}{8}''$ per foot and a small diameter of $\frac{9}{16}''$.



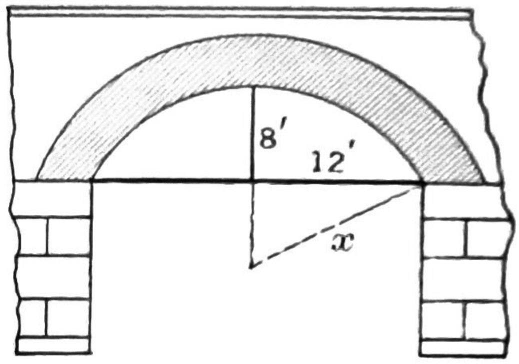
Ex. 665. One method of turning a taper in a lathe is by offsetting the tail stock. The tail stock offset is equal to half of the total taper. Thus in the above figure the set-over for the tail stock is $\frac{1}{8}''$. In general, $S = \dfrac{D - d}{2}$, where S = set-over, D = large diameter, d = small diameter. If the taper is to be turned for only a part of the length of the work, the set-over is half the taper for the entire length of the work. Thus, in the annexed drawing, the set-over will be the same as if the large diameter were AB and the small diameter d. Since $\triangle\ BHG \sim \triangle\ EFG$, we have $EF : FG = BH : GH$ in which EF, FG, GH are known. Hence BH can be found. The set-over equals $BH + EF$. Prove $S = \dfrac{D - d}{2} \times \dfrac{L}{l}$ where



D = large diameter of tapered portion
d = small diameter of tapered portion
L = total length of the piece
l = length of tapered portion
S = set-over.

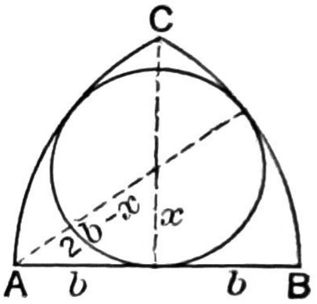
Ex. 666. Find the tail stock offset for turning a taper on a piece of work 10″ long, if the tapered portion is $4\frac{1}{2}$″ long and measures $1\frac{3}{8}$″ in diameter at the large end and $\frac{13}{16}$″ at the small end.

Ex. 667. In the segmental arch here shown, the span is 24′ and the rise is 8′. Find the radius.



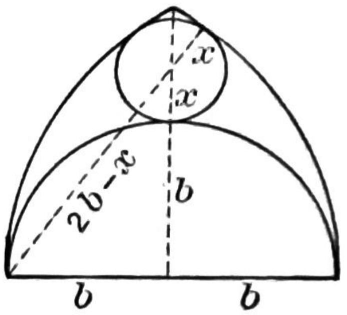
Ex. 668. Inscribe a circle in an equilateral Gothic arch.

Sug. Assume the problem done. Then $x^2 + b^2 = (2\,b - x)^2$, etc.



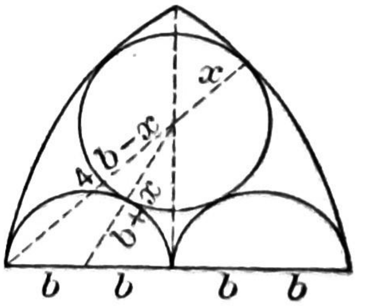
Ex. 669. Draw a circle that shall be tangent to the equilateral Gothic arch and to the semicircle.

Sug. $b^2 + (b + x)^2 = (2\,b - x)^2$, etc.



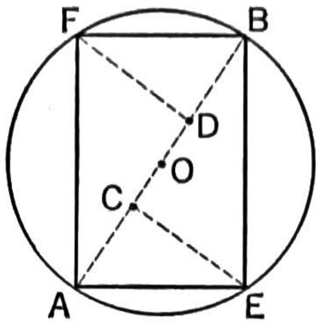
Ex. 670. Draw a circle that shall be tangent to the equilateral Gothic arch and to the two semicircles.

Sug. $(4\,b - x)^2 - 4\,b^2 = (b + x)^2 - b^2$.



Note: The three above constructions occur frequently in church window designs.

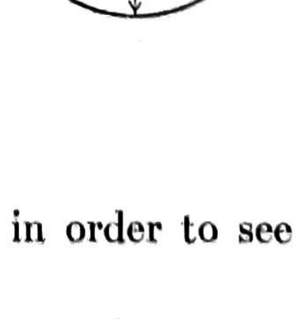
Ex. 671. The strongest rectangular beam that can be cut from a round log is one in which the depth = width $\times\sqrt{2}$. If a diameter AB is trisected at C and D, and if $CE \perp AB$ and $DF \perp AB$, prove $AFBE$ is a rectangle and that $AF = FB \times \sqrt{2}$.



Ex. 672. What are the dimensions of the strongest rectangular beam that can be cut from a log whose diameter is 18″?

Ex. 673. Show that the distance that is visible from a height h ft. above the surface of the earth is approximately $\sqrt{\tfrac{3}{2}h}$ miles.

Solution.



$$x^2 = \left(8000 + \frac{h}{5280}\right)\frac{h}{5280}$$

$$= \tfrac{800}{528}\,h + \frac{h^2}{5280^2}$$

$= \tfrac{800}{528}\,h$, neglecting the second term because it is negligibly small in comparison with the first

$$= \tfrac{100}{66}\,h$$

$= \tfrac{3}{2}\,h$, approximately

$$\therefore x = \sqrt{\tfrac{3}{2}h}$$

Ex. 674. How high must an aviator rise in order to see 75 miles over the ocean?

Ex. 675. Two railroad tracks intersect at 90°. To a passenger in one train traveling at the rate of 32 miles an hour the other seems to have a velocity of 40 miles an hour. Find its actual velocity.

Ex. 676. Find the resultant of two forces, each of 10 pounds, acting at an angle of 60°. (See Ex. 200.)

Ex. 677. Find the resultant of two forces P and $(P + Q)$ acting at 120°.

Ex. 678. Resolve a force of 20 pounds into two equal components, one of them making an angle of 45° with the force.

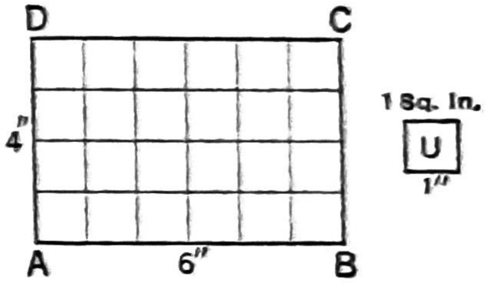
Ex. 679. If in Ex. 424 the width of the sewer (AB) is w feet, prove that its depth (EF) is $\frac{w}{2}(4 - \sqrt{2})$ feet.

Ex. 680. If in Ex. 424 the depth of the sewer is d feet prove that its width is $\frac{d(4 + \sqrt{2})}{7}$ feet.

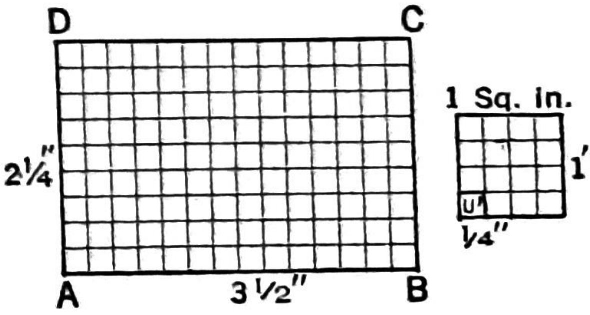
BOOK IV

AREAS OF POLYGONS

271. The space inclosed by the boundary of a plane figure is called the *area* of the figure. Area is expressed in square inches, square feet, square yards, square miles, square centimeters, etc. Consider the rectangle $ABCD$ whose base is $6''$ and whose altitude is $4''$. It is evident that the inclosed space may be divided into 24 squares, such that the side of each square is $1''$. If we denote one of the squares by U, we see that U is contained 24 times in $ABCD$. We say that the area of $ABCD = 24\ U =$ 24 sq. in. $= (6 \times 4)$ sq. in. Taking the unit of surface as a square whose side is the unit of length, we may define *the area of any surface as the number of times that the unit of surface goes into the given surface.*



If AB is $3\frac{1}{2}''$ and CD is $2\frac{1}{4}''$, we take any number which is a common measure of $3\frac{1}{2}''$ and $2\frac{1}{4}''$, e.g. $\frac{1}{4}''$, and therefore a square $\frac{1}{4}''$ on a side may be used as the unit of surface measure. Therefore rectangle $ABCD$ contains (14×9) squares each equal to U', where U' is $\frac{1}{16}$ of 1 sq. in. Hence area of rectangle $ABCD$ $= 126\ U' = (\frac{126}{16})$ sq. in. $= (\frac{63}{8})$ sq. in. $= (3\frac{1}{2} \times 2\frac{1}{4})$ sq. in. We thus see that in both cases the area of the



rectangle equals the product of its base and altitude. If AB and AD have no common measure, by taking the unit square sufficiently small we may approximate as close as we please to the area of the rectangle.

272. We shall therefore assume that in all cases *the area of a rectangle is equal to the product of its base and altitude.*

273. The base and altitude of a rectangle are called its *dimensions.*

274. Any two rectangles are to each other as the products of their dimensions.

$$\text{For } R = b \cdot a,\ R' = b' \cdot a', \quad \therefore \frac{R}{R'} = \frac{b \cdot a}{b' \cdot a'}.$$

275. Rectangles having equal bases are to each other as their altitudes.

$$\text{For } \frac{R}{R'} = \frac{ba}{b'a'} = \frac{a}{a'}, \text{ (since } b = b'\text{).}$$

276. Rectangles having equal altitudes are to each other as their bases.

$$\text{For } \frac{R}{R'} = \frac{ba}{b'a'} = \frac{b}{b'}, \text{ (since } a = a'\text{).}$$

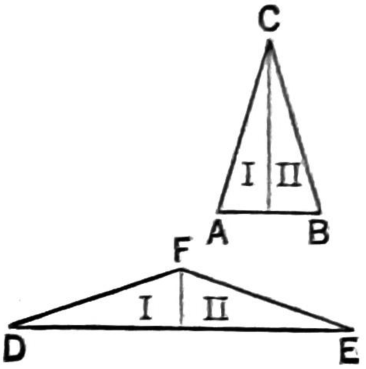
Ex. 681. Find the area of a walk 2′ 6″ wide which surrounds a lot 250′ by 400′, if the dimensions of the lot are taken on the inside of the walk.

Ex. 682. The dimensions of a rectangle are in the ratio 3 : 4, and its area is 588 sq. yds. Find the dimensions.

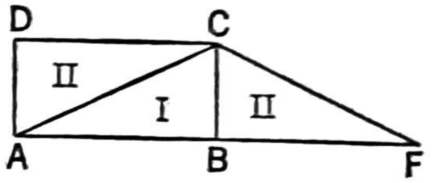
Ex. 683. By drawing a line parallel to PQ, divide rectangle $PQRS$ into two rectangles whose areas are to each other as 1 : 2.

Ex. 684. The length of a rectangle is $3\frac{1}{2}'$ more than its width, and the diagonal is $6\frac{1}{2}'$. Find the area of the rectangle.

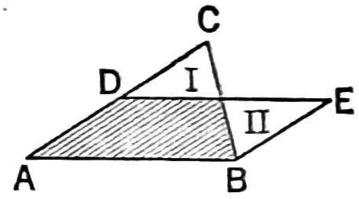
277. Polygons that have equal areas are called *equivalent* polygons. Thus if isosceles $\triangle ABC$ is cut into two congruent $\triangle$s and these $\triangle$s are placed so as to form $\triangle DEF$,



then $\triangle DEF$ covers the same extent of surface as $\triangle ABC$. (Each is composed of $\triangle I + \triangle II$.) We say that $\triangle DEF$ is equivalent to $\triangle ABC$. Note that equivalent figures are not necessarily congruent, but congruent figures are always equivalent.

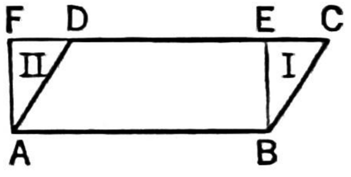


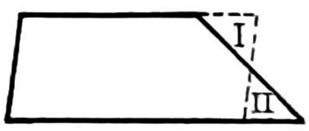
If the rectangle $ABCD$ is divided into two congruent triangles by the diagonal AC, and if $\triangle II$ is placed in the position BCF, then rectangle $ABCD$ is equivalent to $\triangle ACF$ (since each is composed of $\triangle I + \triangle II$).



Ex. 685. In the adjoining figure, D is the midpoint of side AC of $\triangle ABC$, $DE \parallel AB$ and $BE \parallel AC$. Prove $\triangle ABC$ is equivalent to parallelogram $ABED$.

278. To *transform* a given polygon is to construct a second polygon equivalent to the given one. Thus parallelogram $ABCD$ is transformed into a rectangle by drawing BE and AF each perpendicular to CD. It can readily be shown that $ABEF$ is a rectangle, and that $\triangle I \cong \triangle II$. $\therefore ABCF - \triangle II = ABCF - \triangle I$. $\therefore$ parallelogram $ABCD$ = rectangle $ABEF$. (Here the sign of equality should be read "is equivalent to.")

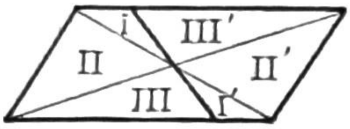




Ex. 686. Transform a given trapezoid into a parallelogram.

Sug. Through the midpoint of one leg draw a line parallel to the other leg.

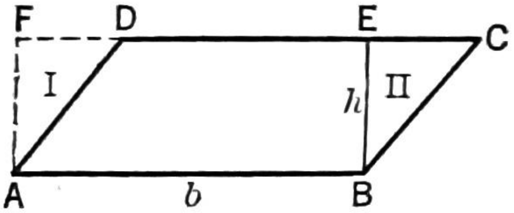
Ex. 687. Prove that any line drawn through the intersection of the diagonals of a parallelogram and terminated in the opposite sides bisects the parallelogram (i.e. cuts it into two equivalent parts).



AREA OF A PARALLELOGRAM

Proposition I. Theorem

279. The area of a parallelogram is equal to the product of its base and its altitude.



Given ▱ $ABCD$, with its base AB denoted by b and its altitude BE by h.

Prove area of ▱ $ABCD = b \times h$.

Plan is to transform ABCD into a rectangle (see 278).

Proof. Draw $AF \perp CD$. Then $BEFA$ is a rectangle. (Why?)

(1) $AD = BC$	(1) 84.
(2) $AF = BE$	(2) 84.
(3) $\therefore \triangle I \cong \triangle II$	(3) 75.
(4) $\therefore ABCF - \triangle I = ABCF - \triangle II$	(4) Ax. 3.
(5) $\therefore$ ▱ $ABCD$ = rectangle $ABEF$	(5) Sub.
(6) but area rectangle $ABEF = b \times h$	(6) 272.
(7) $\therefore$ area ▱ $ABCD = b \times h$	(7) Sub.

Q.E.D.

280. *Cor.* 1. Parallelograms having equal bases and equal altitudes are equivalent.

281. *Cor.* 2. Any two parallelograms are to each other as the products of their bases and altitudes.

282. *Cor.* 3. Parallelograms having equal bases are to each other as their altitudes.

283. *Cor.* 4. Parallelograms having equal altitudes are to each other as their bases.

Sug. For proof of above corollaries use method employed in 274.

Ex. 688. Two adjacent sides of a ▱ are 9″ and 14″, and their included angle is 60°. Find the area of the ▱.

Ex. 689. Find the area of a ▱ that has two adjacent sides 6′ and 18′, and their included angle 45°.

GROUP XV

Complete each of the following statements:

(*a*) A parallelogram whose base is 8″ is equivalent to a square whose side is 4″. The altitude of the parallelogram is . . . .

(*b*) If the diagonal of a square is $5\sqrt{2}$, its area is . . . .

(*c*) The area of one square is double that of another square. The ratio of their sides is . . . .

(*d*) The square constructed on the diagonal of a given square is . . . the given square.

(*e*) If a rectangle and a parallelogram are equivalent and have equal bases, they must also have equal . . . .

(*f*) If the base of a parallelogram is doubled and the area is unaltered, then the altitude is . . . .

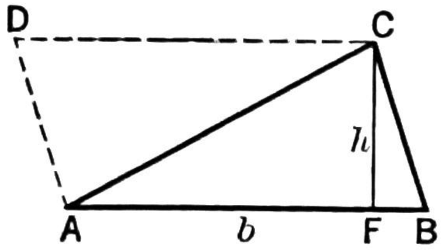
(*g*) If a square and a parallelogram are equivalent, the side of the square is . . . between the base and altitude of the parallelogram.

(*h*) As the angle between two adjacent sides of a parallelogram increases, the area of the parallelogram . . . ; but as the angle becomes . . . the area . . . .

AREA OF A TRIANGLE

Proposition II. Theorem

284. The area of a triangle is equal to one half the product of its base and altitude.



Given △ ABC, with its base AB denoted by b and its altitude CF denoted by h.

Prove area of △ $ABC = \frac{1}{2} b \times h$.

Analysis. $b \times h$ *suggests the area of a* ▱. *Hence we must prove that* △ ABC *is half of a* ▱ *whose base is* b *and whose altitude is* h.

Proof. Through A draw a line ∥ BC, and through C draw a line ∥ AB, and let these lines meet at D, thus forming ▱ $ABCD$. Then

(1) △ $ABC = \frac{1}{2}$ ▱ $ABCD$	(1) 85.
(2) but ▱ $ABCD = b \times h$	(2) 279.
(3) ∴ △ $ABC = \frac{1}{2} b \times h$	(3) Sub.

Q.E.D.

285. *Cor.* 1. Triangles having equal bases and equal altitudes are equivalent.

286. *Cor.* 2. Any two triangles are to each other as the products of their bases and altitudes.

287. *Cor.* 3. Triangles having equal bases are to each other as their altitudes.

288. *Cor.* 4. Triangles having equal altitudes are to each other as their bases.

289. *Cor.* 5. If two triangles have a common base and their vertices lie in a line parallel to the base, the triangles are equivalent.

Exercises

Ex. 690. Find the area of an equilateral triangle whose side is 6″.

Ex. 691. Prove that the area of an equilateral triangle whose side is b equals $\frac{b^2}{4}\sqrt{3}$.

Ex. 692. Find the area of a triangle whose base is 16″ and whose base angles are 120° and 30°.

Ex. 693. ABC is a triangle inscribed in a ⊙ whose radius is 5′. AB is the diameter of the ⊙ and $\angle ABC = 30°$. Find the area of the triangle.

Ex. 694. The base of an isosceles triangle is 20″. A base angle is 30°. Find the area of the triangle.

Ex. 695. Transform a given triangle into a right triangle.

Sug. 289 is used for most transformations.

Ex. 696. What is the locus of the vertices of all the equivalent triangles constructed on the same base?

Ex. 697. Prove that the diagonals of a parallelogram divide it into four equivalent triangles.

Ex. 698. The base of a triangle is 12″ and its altitude is 5″. Find the area of the trapezoid cut off by a line 3″ from the vertex.

Ex. 699. Prove that the area of a rhombus equals half the product of its diagonals.

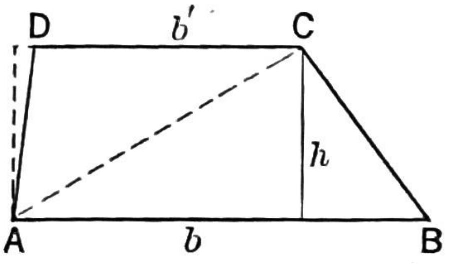
Sug. The diagonals of a rhombus bisect each other at right angles.

Ex. 700. One side of a rhombus is 20′, and its longer diagonal is 32′. Find the area of the rhombus.

AREA OF A TRAPEZOID

Proposition III. Theorem

290. The area of a trapezoid is equal to one half the product of its altitude and the sum of its bases.



Given trapezoid $ABCD$, with bases denoted by b and b', and altitude by h.

Prove area of trapezoid $ABCD = \frac{1}{2}h(b + b')$.

Plan is to divide the trapezoid into two $\triangle$s *and sum their areas.*

Proof. Draw AC. Then

(1) area $\triangle ABC = \frac{1}{2}bh$	(1) 284.
(2) area $\triangle ACD = \frac{1}{2}b'h$	(2) 284.
(3) $\therefore \triangle ABC + \triangle ACD = \frac{1}{2}bh + \frac{1}{2}b'h$	(3) Ax. 2.
(4) $\therefore$ area $ABCD = \frac{1}{2}h(b + b')$	(4) Sub., and factoring.

Q.E.D.

Exercises

Ex. 701. The bases of a trapezoid are 8″ and 15″, and the altitude is 6″. Find the area of the trapezoid.

Ex. 702. The area of a trapezoid is 216 sq. ft. Its altitude is 9′, and one base is 28′. Find the other base.

Ex. 703. The parallel sides of a trapezoid are 125′ and 185′ respectively, and the other sides are each 50′. Find the area of the trapezoid.

Ex. 704. The bases of a trapezoid are 12″ and 30″ respectively, and each lower base angle is 60°. Find the area of the trapezoid correct to one decimal place.

Ex. 705. Prove that the area of an isosceles right triangle equals one fourth the square of the hypotenuse.

Ex. 706. Prove by areas that the product of the legs of a right triangle is equal to the product of the hypotenuse and the altitude upon the hypotenuse.

Ex. 707. Prove that the area of a trapezoid is equal to the product of its median and its altitude.

Ex. 708. Two triangles are equivalent if two sides of the one are equal respectively to two sides of the other, and the included angles are supplementary.

Ex. 709. Divide a triangle into three equivalent parts by two lines passing through a vertex of the triangle.

Ex. 710. Transform a given triangle into an isosceles triangle having the same base.

Ex. 711. Transform a given triangle into a right triangle, having the hypotenuse equal to one side of the given triangle.

Ex. 712. Transform a given trapezoid into a triangle.

Ex. 713. Transform an equilateral triangle into a rectangle.

Ex. 714. Divide a trapezoid into two equivalent parts by a line perpendicular to the bases.

Ex. 715. If P is any point in the side BC of $\square\, ABCD$, prove $\triangle\, PAB + \triangle\, PCD = \frac{1}{2} \square\, ABCD$.

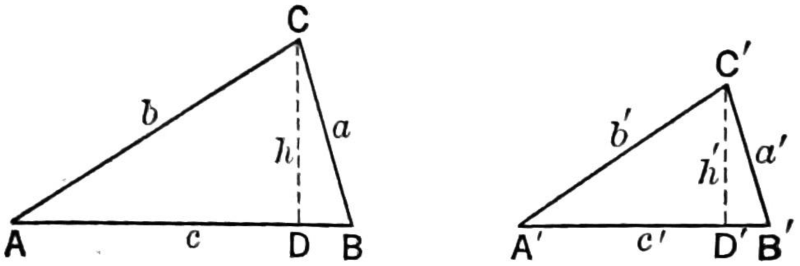
Ex. 716. If D and E are the midpoints of sides BC and AC respectively of $\triangle\, ABC$, prove that $\triangle\, ABD$ is equivalent to $\triangle\, ABE$.

Ex. 717. The bases of a trapezoid are 16′ and 10′ respectively, and each lower base angle is 45°. Find the altitude of an equivalent triangle whose base is 12′.

AREAS OF SIMILAR TRIANGLES

Proposition IV. Theorem

291. The areas of two similar triangles are to each other as the squares of any two corresponding sides.



Given $\triangle ABC \sim \triangle A'B'C'$.

Prove $\dfrac{\text{area } \triangle ABC}{\text{area } \triangle A'B'C'} = \dfrac{c^2}{c'^2} = \dfrac{a^2}{a'^2} = \dfrac{b^2}{b'^2}$

Plan is to use 286.

Proof. Draw the corresponding altitudes h and h'. Then

(1) $\dfrac{\triangle ABC}{\triangle A'B'C'} = \dfrac{c \times h}{c' \times h'}$	(1) 286.
(2) $\therefore \dfrac{\triangle ABC}{\triangle A'B'C'} = \dfrac{c}{c'} \times \dfrac{h}{h'}$	(2) Writing $\dfrac{c \times h}{c' \times h'}$ as a product of two fractions.
(3) but $\dfrac{h}{h'} = \dfrac{c}{c'}$	(3) Ex. 456.
(4) $\therefore \dfrac{\triangle ABC}{\triangle A'B'C'} = \dfrac{c}{c'} \times \dfrac{c}{c'}$	(4) Sub.
(5) $\therefore \dfrac{\triangle ABC}{\triangle A'B'C'} = \dfrac{c^2}{c'^2}$	(5) Simplifying.
(6) but $\dfrac{c}{c'} = \dfrac{a}{a'} = \dfrac{b}{b'}$	(6) Corr. sides of $\sim$ polygons are in proportion.

(7) $\therefore \frac{c^2}{c'^2} = \frac{a^2}{a'^2} = \frac{b^2}{b'^2}$	(7) Squares of equals are equal.
(8) $\therefore \frac{\triangle ABC}{\triangle A'B'C'} = \frac{c^2}{c'^2} = \frac{a^2}{a'^2} = \frac{b^2}{b'^2}$	(8) Sub.

Q.E.D.

Note: According to this proposition, if b is three times b', then the area of $\triangle ABC$ is nine times the area of $\triangle A'B'C'$.

Exercises

Ex. 718. What is the ratio of the areas of two equilateral triangles whose sides are 5″ and 20″ respectively?

Ex. 719. Construct a triangle four times as large as a given triangle and similar to it.

Ex. 720. The base of a triangle is 15′, its area is 60 sq. ft. Find the area of a similar triangle whose altitude is 6′.

Ex. 721. The base of a certain triangle is 16′. Find the base of a similar triangle whose area is $4\frac{1}{2}$ times that of the former.

Ex. 722. The areas of two similar triangles are to each other as the squares of their perimeters.

Ex. 723. The areas of two similar triangles are to each other as the squares of corresponding altitudes, or as the squares of corresponding medians, or as the squares of corresponding angle bisectors.

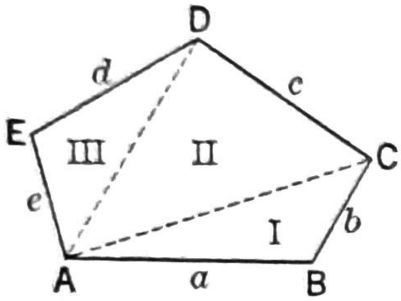
Ex. 724. The corresponding sides of two similar triangles are in the ratio 5 : 9. The sum of their areas is 636 sq. ft. Find the area of each triangle.

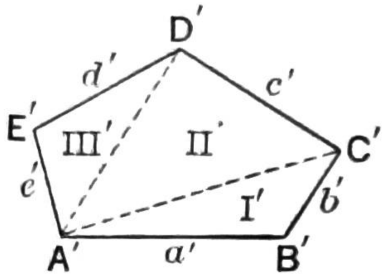
Ex. 725. The side of an equilateral triangle is b. The side b is prolonged a distance x such that the equilateral triangle constructed on $(b + x)$ as a side is double the area of the given triangle. Find x.

AREAS OF SIMILAR POLYGONS

Proposition V. Theorem

292. The areas of two similar polygons are to each other as the squares of any two corresponding sides.





Given polygon $ABCDE \sim$ polygon $A'B'C'D'E'$, with respective areas denoted by S and S', and corresponding sides denoted by a and a'.

Prove $\frac{S}{S'} = \frac{a^2}{a'^2}$.

Plan is to divide the polygons into pairs of similar triangles, and then apply 291.

Proof. From any two corresponding vertices A and A' draw all the possible diagonals. Then

(1) $\triangle I \sim \triangle I'$, $\triangle II \sim \triangle II'$, $\triangle III \sim \triangle III'$	(1) 260.
(2) $\therefore \frac{\triangle I}{\triangle I'} = \frac{a^2}{a'^2}$ $\frac{\triangle II}{\triangle II'} = \frac{c^2}{c'^2}$ $\frac{\triangle III}{\triangle III'} = \frac{d^2}{d'^2}$	(2) 291.
(3) but $\frac{a}{a'} = \frac{c}{c'} = \frac{d}{d'}$	(3) Corr. sides of $\sim$ polygons are in proportion.

(4) $\therefore \frac{a^2}{a'^2} = \frac{c^2}{c'^2} = \frac{d^2}{d'^2}$	(4) Squares of equals are equal.
(5) $\therefore \frac{\triangle I}{\triangle I'} = \frac{\triangle II}{\triangle II'} = \frac{\triangle III}{\triangle III'}$	(5) Ax. 1.
(6) $\therefore \frac{\triangle I + \triangle II + \triangle III}{\triangle I' + \triangle II' + \triangle III'} = \frac{\triangle I}{\triangle I'}$	(6) 219.
(7) $\therefore \frac{S}{S'} = \frac{\triangle I}{\triangle I'}$	(7) Sub.
(8) $\therefore \frac{S}{S'} = \frac{a^2}{a'^2}$	(8) Ax. 1.

Q.E.D.

293. *Cor.* The areas of two similar polygons are to each other as the squares of their perimeters.

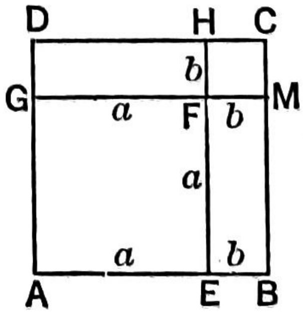
Exercises

Ex. 726. One side of a polygon is 5″. Find the corresponding side of a similar polygon whose area is twice that of the first polygon.

Ex. 727. The areas of two similar polygons are in the ratio $m : n$. What is the ratio of any two corresponding sides?

Ex. 728. The area of a polygon is 500 sq. ft., and one of its sides is 20′. Find the corresponding side of a similar polygon whose area is 360 sq. ft.

294. If a and b are two given lines, then $a \times b$ may be represented geometrically by a rectangle whose dimensions are a and b. Some of the well-known algebraic identities can be proved geometrically. Thus to prove geometrically that $(a + b)^2 = a^2 + 2\,ab + b^2$ we let $AE = a$, $EB = b$. On AB as a side, construct square $ABCD$. On AE as a side, con-



struct square $AEFG$. Prolong EF till it meets CD at H. Prolong GF till it meets BC at M. Then we have

$$\text{square } ABCD = \text{square } AEFG + \text{square } CHFM + \text{rectangle } EBMF + \text{rectangle } GFHD$$

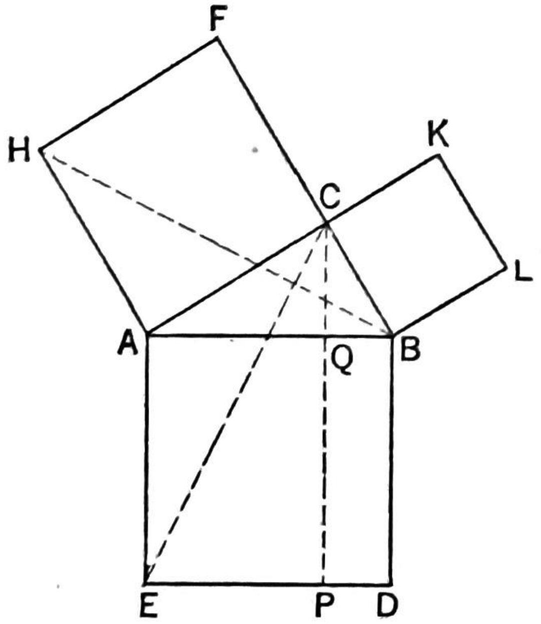
$$\therefore (a + b)^2 = a^2 + b^2 + ab + ab = a^2 + 2\,ab + b^2$$

295. Referring to 254, the equation $a^2 = cx$ may be interpreted to mean that the square on a is equivalent to the rectangle whose dimensions are c and x; $b^2 = c(c - x)$ means that the square on b is equivalent to the rectangle whose dimensions are c and $c - x$. This suggests that the Pythagorean Theorem can be proved by areas (see 296).

PYTHAGOREAN THEOREM

Proposition VI. Theorem

296. The square on the hypotenuse of a right triangle is equivalent to the sum of the squares on the other two sides.



Given right $\triangle ABC$, with AD the square on the hypotenuse AB, and AF and BK the squares on the legs.

Prove square AD = square AF + square BK.

Plan is to divide square AD into two rectangles which are respectively equivalent to AF and BK. This can be done by drawing the altitude on the hypotenuse, and prolonging it (*see* 295).

Proof. Draw $CQ \perp AB$, and prolong it till it meets ED at P. Draw BH and CE.

(1) BCF is a straight line	(1) $\angle BCF$ is a straight $\angle$.
(2) $\therefore$ square $AF = 2 \triangle AHB$	(2) They have the same base and equal altitudes.
(3) $CP \parallel AE$	(3) 55.
(4) $\therefore$ rectangle $AP = 2 \triangle ACE$	(4) Same reason as (2).
(5) but $\triangle AHB \cong \triangle ACE$ (for $AH = AC$ $AB = AE$ $\angle HAB = \angle CAE$)	(5) s. a. s. = s. a. s. (sides of a square are equal. Each equals $\angle BAC + 90°$)
(6) $\therefore$ square AF = rectangle AP	(6) Ax. 1.

In like manner, by drawing AL and CD, we can prove that square BK = rectangle BP.

(7) $\therefore$ square AF + square BK = rect. AP + rect. BP	(7) Ax. 2.
(8) $\therefore$ square AF + square BK = square AD	(8) Sub.

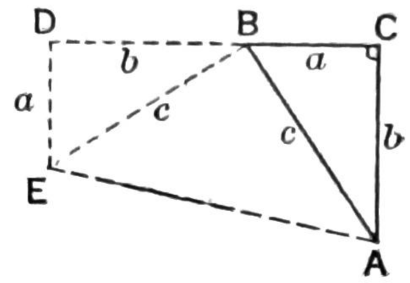
Q.E.D.

297. *Cor.* The square on one leg of a right triangle is equivalent to the difference between the square on the hypotenuse and the square on the other leg.

298. The above proof (296) was first given by Euclid, a famous Greek mathematician, about 300 B.C.

Exercises

Ex. 729. Prove the Pythagorean Theorem by prolonging CB to D so that $BD = b$ (see annexed drawing), then drawing $DE \perp CD$ and of length a, then drawing EB and EA. (Proof based on this construction was first given by President James A. Garfield.)



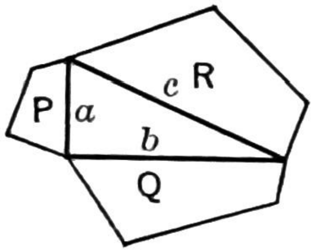
Sug. Trapezoid $ACDE = \triangle ABC + \triangle ABE + \triangle BDE$.

Ex. 730. If similar polygons are constructed on the sides of a right triangle as corresponding sides, prove that the polygon on the hypotenuse is equivalent to the sum of the polygons on the other two sides.

Solution. $\frac{P}{R} = \frac{a^2}{c^2}, \frac{Q}{R} = \frac{b^2}{c^2}$

$$\therefore \frac{P + Q}{R} = \frac{a^2 + b^2}{c^2} = 1$$

$$\therefore P + Q = R$$



Ex. 731. Construct a square equivalent to the sum of two given squares.

Sug. Make use of 296.

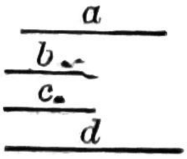
Ex. 732. Construct a square equivalent to the difference of two given squares.

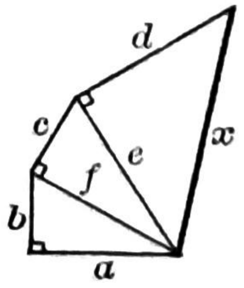
Sug. Make use of 297.

Ex. 733. Construct a square equivalent to the sum of any number of given squares.

Indicated solution. Let a, b, c, d be sides of four given squares. Then x is the side of the required square. For,

$$\begin{aligned} x^2 &= d^2 + e^2 \\ &= d^2 + c^2 + f^2 \\ &= d^2 + c^2 + b^2 + a^2 \end{aligned}$$

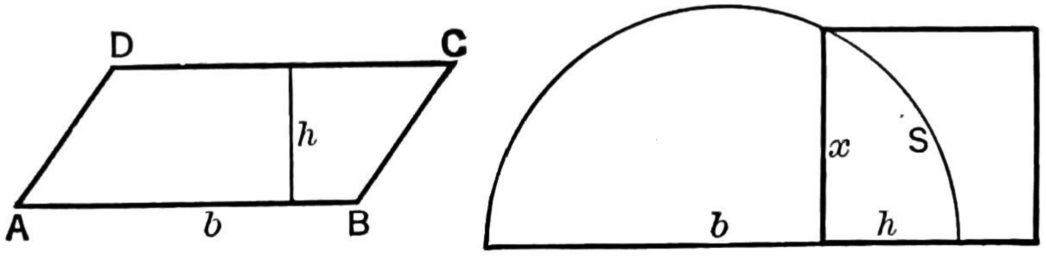




EQUIVALENT POLYGONS

Proposition VII. Problem

299. To construct a square equivalent to a given parallelogram.



Given parallelogram $ABCD$ with base b and altitude h.

Required to construct a square equivalent to $ABCD$.

Analysis. Assume the problem done, and let x be the side of the required square. Then $x^2 = bh$, i.e. $b : x = x : h$, suggesting that x is the mean proportional between b and h.

Construction. Draw x, the mean proportional between b and h. On x as a side draw square S. Then this is the required square.

Proof.

(1) $b : x = x : h$	(1) Const.
(2) $\therefore x^2 = bh$	(2) 213.
(3) but x^2 = area of S and bh = area of $ABCD$	(3) 279.
(4) Area S = area $\square\ ABCD$	(4) Sub. Q.E.D.

Ex. 734. Construct a square equivalent to a given triangle.

Sug. $x^2 = \frac{1}{2} bh$. Hence either $\frac{1}{2} b : x = x : h$ or $\frac{1}{2} h : x = x : b$

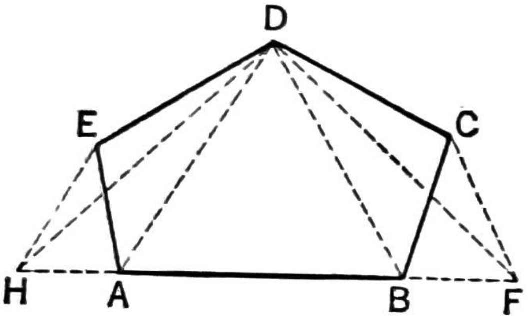
Ex. 735. Construct a square equivalent to $\frac{2}{3}$ of a given triangle.

Ex. 736. Construct a square equivalent to five times a given square.

Ex. 737. Construct a square equivalent to a given trapezoid.

Proposition VIII. Problem

300. To transform a given polygon into a triangle.



Given polygon $ABCDE$.

Required to transform $ABCDE$ into a triangle.

Plan is to transform ABCDE into an equivalent 4-sided figure, and then to transform the latter into a triangle. 289 *is the means used to accomplish this.*

Construction and proof. (*a*) Draw DB.

(*b*) Through C draw a line $\parallel DB$ meeting AB produced at F.

(*c*) Draw DF.

The polygon $ABCDE$ has now been transformed into $AFDE$ (which has one less side than $ABCDE$).

For, $\triangle BDC = \triangle BDF$ (289)

$ABDE = ABDE$ (Iden.)

$\therefore ABDE + \triangle BDC = ABDE + \triangle BDF$ (Ax. 2)

$\therefore ABCDE = AFDE$ (Sub.)

We now proceed to transform $AFDE$ into a triangle, and to avoid making the drawing appear too complicated, it is advisable to work with the left side of the given polygon.

(*d*) Draw DA.

(*e*) Through E draw a line $\parallel DA$ meeting BA produced at H.

(*f*) Draw DH.

The quadrilateral $AFDE$ has now been transformed into $\triangle DFH$.

For, $\triangle ADE = \triangle ADH$ (289)

$\triangle ADF = \triangle ADF$ (Iden.)

$\therefore \triangle ADF + \triangle ADE = \triangle ADF + \triangle ADH$ (Ax. 2)

$\therefore AFDE = \triangle DFH$ (Sub.)

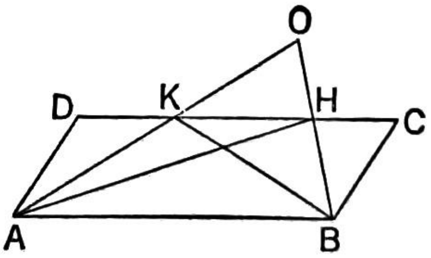
$\therefore ABCDE = \triangle DFH$ (Ax. 1)

Q.E.D.

Exercises

Ex. 738. Construct a square equivalent to a given polygon.

Sug. First transform the given polygon into a triangle, and then construct a square equivalent to the triangle.

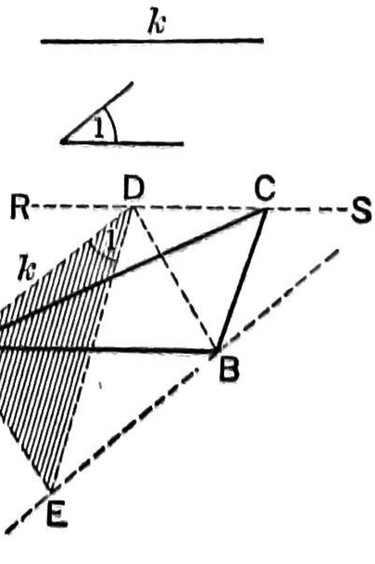


Ex. 739. In the adjoining figure, $ABCD$ is a parallelogram. Prove $\triangle OAH$ is equivalent to $\triangle OBK$.

Ex. 740. In $\triangle ABC$ the medians AE and CD intersect at O. Prove $\triangle AOC$ is equivalent to the quadrilateral $DBEO$.

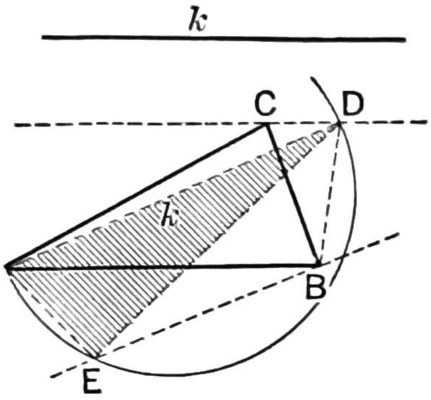
Ex. 741. Transform $\triangle ABC$ into another triangle having a side equal to a given line k, and one adjacent angle equal to a given angle.

Solution. We shall use two successive applications of 289. First construct $\triangle ABD$ with side $AD = k$. Hence $\triangle ADB = \triangle ABC$. Next construct



$\angle ADE = \angle 1$. Hence $\triangle ADE$ is the required triangle (shaded in the drawing). For, $\triangle ADE = \triangle ADB = \triangle ABC$.

Ex. 742. Transform $\triangle ABC$ into a right triangle having the hypotenuse equal to a given line k.



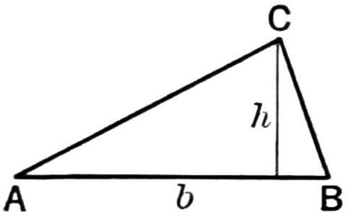
The adjoining drawing gives the solution based on two successive applications of 289.

The problem might also have been approached by algebraic analysis. Thus: Assume the problem done.

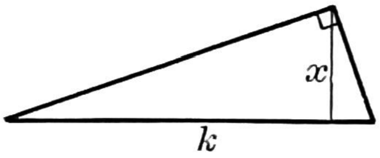
Then $\frac{1}{2}kx = \frac{1}{2}bh$

$\therefore kx = bh$

$\therefore k : b = h : x$



Thus x is found by constructing the fourth proportional to k, b, and h. The problem then reduces to constructing a right triangle when the hypotenuse and the altitude upon the hypotenuse are known.



Miscellaneous Theorems

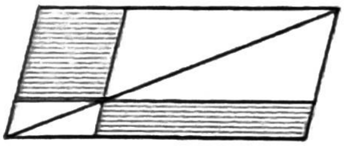
Ex. 743. P is any point on the median CM of $\triangle ABC$. Prove $\triangle ACP$ is equivalent to $\triangle BCP$.

Ex. 744. The line joining the midpoints of two sides of a triangle cuts off a triangle whose area is one fourth of the original triangle.

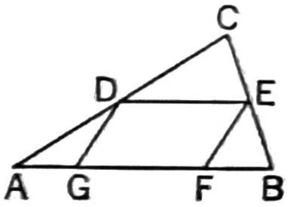
Ex. 745. ABC is a triangle whose medians meet at O. Prove $\triangle AOB$ is one third of $\triangle ABC$.

Ex. 746. The lines joining the midpoint of one diagonal of a quadrilateral to the opposite vertices divide the quadrilateral into two equivalent parts.

Ex. 747. If P is any point in the diagonal AC of parallelogram $ABCD$, prove $\triangle ABP$ is equivalent to $\triangle APD$.



Ex. 748. If through any point on the diagonal of a parallelogram lines are drawn parallel to its sides, the parallelograms on opposite sides of the diagonal are equivalent.



Ex. 749. In the adjoining figure *DE* joins the midpoints of *AC* and *BC*; $DG \parallel EF$. Prove $DEFG = \frac{1}{2} \triangle ABC$.

Ex. 750. *ABCD* is a parallelogram, *E* is any point in *CD*, *F* is any point in *AD*. Prove $\triangle ABE$ is equivalent to $\triangle BCF$.

Ex. 751. If from the midpoint of the base of a triangle lines parallel to the sides are drawn, the parallelogram thus formed is equivalent to one half the triangle.

Ex. 752. If diagonal *AC* of quadrilateral *ABCD* bisects diagonal *BD*, then $\triangle ABC$ is equivalent to $\triangle ADC$.

Ex. 753. Two isosceles triangles are equivalent if their legs are equal each to each, and the altitude of one is equal to half the base of the other.

Ex. 754. The line joining any vertex of a parallelogram to the midpoint of one of the non-adjacent sides divides the parallelogram into a triangle and a trapezoid such that the trapezoid is three times as large as the triangle.

Ex. 755. Lines drawn from any point in one side of a triangle to the midpoints of the other two sides form a quadrilateral which is equivalent to one half the given triangle.

Ex. 756. The area of any polygon is equal to half the product of its perimeter and the radius of the inscribed circle.

Sug. Draw lines from center of circle to vertices of polygon.

Ex. 757. The areas of two similar parallelograms are to each other as the products of their diagonals.

Ex. 758. If on the median of a trapezoid a triangle is constructed whose vertex is in the upper base of the trapezoid, the triangle is equivalent to one fourth the trapezoid.

Ex. 759. If the midpoints of two adjacent sides of a parallelogram are joined, a triangle is formed which is equivalent to one eighth of the parallelogram.

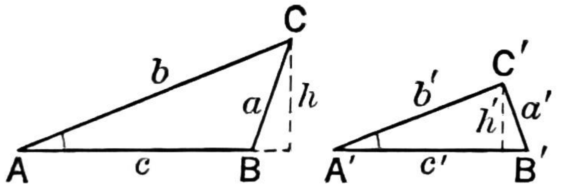
Ex. 760. The diagonals *AC* and *BD* of the trapezoid *ABCD* intersect at *O*. If *AB* is either base, prove $\triangle AOD$ is equivalent to $\triangle BOC$.

Ex. 761. If *P* is any point in the side *BC* of parallelogram *ABCD*, and *DP* is drawn meeting *AB* produced at *F*, prove $\triangle ABP$ is equivalent to $\triangle CPF$.

Sug. Compare $\triangle CDF$ with ▱ *ABCD*; also compare ($\triangle PCD + \triangle PAB$) with ▱ *ABCD*.

Ex. 762. If any point within a parallelogram is joined to the four vertices, the sum of either pair of triangles having parallel bases is equivalent to half the parallelogram.

Ex. 763. If two triangles have an angle of one equal to an angle of the other, their areas are to each other as the products of the sides including the equal angles.



Outline of proof.

$$\frac{\triangle ABC}{\triangle A'B'C'} = \frac{ch}{c'h'} = \frac{c}{c'}\cdot\frac{h}{h'} = \frac{c}{c'}\cdot\frac{b}{b'} = \frac{bc}{b'c'}$$

Ex. 764. In $\triangle ABC$, *M* is the midpoint of *AB*, *P* is any point in *AB* between *A* and *M*. If *MD* is drawn parallel to *PC*, meeting *BC* at *D*, prove $\triangle BPD$ is equivalent to half the $\triangle ABC$.

Sug. Draw *CM*, and prove $\triangle BPD = \triangle BCM$.

Ex. 765. Every straight line drawn through the midpoint of the median of a trapezoid cutting both bases divides the trapezoid into two equivalent parts.

Ex. 766. If the midpoints of the sides of any quadrilateral are joined in succession, the parallelogram thus formed is equivalent to half the quadrilateral.

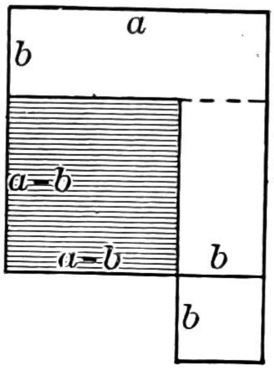
Ex. 767. If two straight lines are drawn from the mid-point of either leg of a trapezoid to the opposite vertices, the triangle thus formed is equivalent to half the trapezoid.

Ex. 768. The area of a trapezoid is equal to the product of one of the legs by the distance from this leg to the mid-point of the other leg.

Sug. Apply Ex. 686.

Ex. 769. If a and b are any two lines (where $a > b$), prove geometrically that $(a - b)^2 = a^2 - 2\,ab + b^2$.

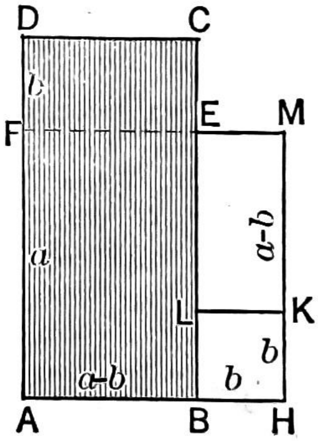
Solution. See adjoining diagram. The square on $(a - b)$ is equivalent to the whole figure diminished by the two rectangles whose dimensions are a and b.



Ex. 770. If a and b are any two lines (where $a > b$), prove geometrically that $(a + b)(a - b) = a^2 - b^2$.

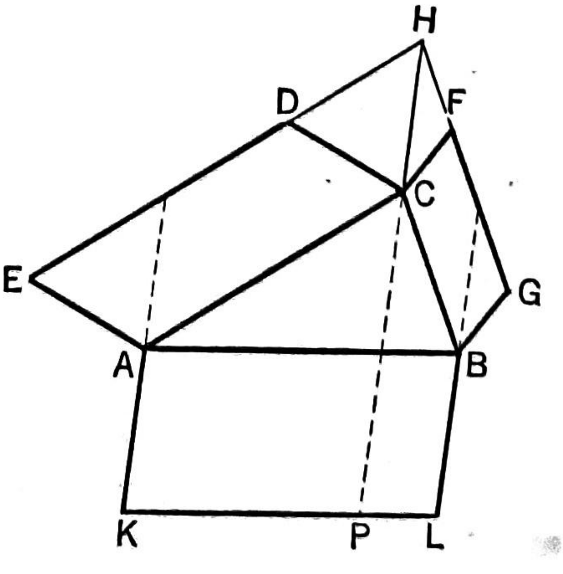
Solution. See adjoining diagram.

$$\begin{aligned} ABCD &= ABEF + FDCE \\ &= ABEF + LKME \\ &= AHMF - BHKL \\ &= a^2 - b^2 \end{aligned}$$



Ex. 771. ABC is any triangle. On AC and BC parallelograms AD and BF are constructed. ED and GF are prolonged to meet at H. On AB parallelogram AL is constructed such that AK is equal and parallel to HC. Prove $\square AL = \square AD + \square BF$. (Pappus' Theorem)

Sug. Prove $\square\, AD = \square\, AH = \square\, AP$; etc.



Numerical Exercises

Ex. 772. Find the ratio of the altitudes of two equivalent triangles if the base of one is five times that of the other.

Ex. 773. Find the area of an isosceles right triangle whose hypotenuse is 14″.

Ex. 774. Upon the diagonal of a rectangle 15′ by 36′ a triangle equivalent to the rectangle is constructed. Find its altitude.

Ex. 775. The altitude of a triangle is h. How far from the vertex must a parallel to the base be drawn to cut off a triangle whose area is one third of the given triangle?

Ex. 776. Find the area of an isosceles trapezoid in which the shorter base is 6″, one leg is 10″, and one angle is 30°.

Ex. 777. One angle of a rhombus is 60°. Find the ratio of the area of the rhombus to that of a square whose perimeter is equal to the perimeter of the rhombus.

Ex. 778. Derive a formula for the area of a triangle in terms of its sides.

Solution.

$$A=\frac{1}{2}c\cdot h_c=\frac{1}{2}c\cdot\frac{2}{c}\sqrt{s(s-a)(s-b)(s-c)}=\sqrt{s(s-a)(s-b)(s-c)}$$

(See Ex. 536.)

Ex. 779. Find the area of a triangle whose sides are 9″, 10″, 14″.

Ex. 780. From the formula of Ex. 778 derive a formula for the area of an equilateral triangle in terms of one side.

Ex. 781. The area of an equilateral triangle is $48\sqrt{3}$ sq. in. Find the altitude.

Ex. 782. The altitude of an equilateral triangle is 10″. Find the area.

Ex. 783. A leg of an isosceles trapezoid is 13″ and its projection on the longer base is 5″. The longer base is 17″. Find a side of the equivalent square.

Ex. 784. The area of a rhombus is 96 sq. ft., and one diagonal is 12′. Find the perimeter.

Ex. 785. One of the angles of a rhombus is 60° and the shorter diagonal is 16′. Find the area of the rhombus.

Ex. 786. From a rectangular piece of cardboard 12″ by 14″ an isosceles trapezoid with bases 14″ and 8″ and sides 5″, and a square with side 8″ are cut. What fractional part of the cardboard is wasted? Draw to scale a figure illustrating this problem.

Ex. 787. Two equilateral triangles have a side of one equal to the altitude of the other. Find the ratio of their areas.

Ex. 788. A circle with a 5″ radius is inscribed in a quadrilateral whose perimeter is 48″. Find the area of the quadrilateral.

Ex. 789. The area of a trapezoid is 96 sq. in., its altitude is 6″. Find the length of the median of the trapezoid.

Ex. 790. The sides of a right triangle are in the ratio 3 : 4 : 5, and the altitude upon the hypotenuse is 20′. Find the area.

Ex. 791. The sides of a triangle are 5″, 7″, 10″. Find the area of the triangle formed by joining the midpoints of the sides.

Ex. 792. A rectangle whose base is three times its altitude is inscribed in a circle whose radius is 10′. Find the area of the rectangle.

Ex. 793. A swimming pool is 60′ long, 20′ wide, 3′ deep at one end, and 8′ deep at the other. Find the number of square feet in the bottom of the pool.

Ex. 794. The bases of a trapezoid are 5′ and 26′, the legs 10′ and 17′, and the projections of the legs on the longer base are in the ratio $2\frac{1}{2} : 1$. Find the area of the trapezoid.

Ex. 795. In the parallelogram *ABCD* the altitude on base AB is 16″, $\angle A$ is 45°, and diagonal DB is 34″. Find the area of the parallelogram.

Ex. 796. If the altitude h of a triangle is increased by a length m, how much must be taken from the base b so that the area shall not be altered?

Ex. 797. In a right triangle the segments of the hypotenuse formed by the altitude upon the hypotenuse are 9″ and 4″. Find the area of the triangle.

Ex. 798. If the sides of a triangle are a, b, c, prove that the radius of the inscribed circle equals $\sqrt{\frac{(s-a)(s-b)(s-c)}{s}}$.

Sug. Apply Ex. 756.

Ex. 799. The sides of a triangle are 8″, 15″, 17″. Find the radius of the inscribed circle.

Ex. 800. An equilateral triangle whose base is 5″ is equivalent to a parallelogram whose base is 10″. Find the altitude of the parallelogram.

Ex. 801. A square whose diagonal is 6″ is equivalent to an equilateral triangle. Find a side of the triangle.

Ex. 802. The sides of two equilateral triangles are 5″ and 6″ respectively. Find the side of an equilateral triangle equivalent to their difference.

Ex. 803. The corresponding sides of two similar triangles are 16′ and 30′ respectively. How long is the side of a triangle which is similar to the given triangles and equivalent to their sum?

Ex. 804. What part of a parallelogram is contained between one half of one side and one third of one of the adjacent sides?

Ex. 805. One leg of a right triangle is a, and the altitude upon the hypotenuse is h. Find the area of the triangle.

Ex. 806. The diagonals of a rhombus are r and s. Find (*a*) the area; (*b*) the perimeter; (*c*) the altitude of the rhombus.

Ex. 807. Through the vertex of a triangle whose area is 112 sq. ft. a line is drawn dividing the triangle into two parts, one containing 16 sq. ft. more than the other. If the base of the triangle is 10′, what is the length of each of the segments into which the base is divided?

Ex. 808. The area of a parallelogram is 100 sq. in. How must a line from one vertex divide one of the opposite sides in order that the triangle cut off shall contain 40 sq. in.?

Ex. 809. The sides of a triangle are 4″, 8″, 10″. The largest angle is bisected. Find the area of each triangle thus formed.

Ex. 810. The diagonals of a parallelogram are 12″ and 20″, and their included angle is 120°. Find the area, the perimeter, and the distance between the longer parallel sides.

Ex. 811. If a circle of radius R is circumscribed about a triangle whose sides are a, b, c, show that the area of the triangle equals $\frac{abc}{4R}$.

Sug. See Ex. 544.

Construction Problems

Ex. 812. Transform a parallelogram into a rhombus.

Ex. 813. Divide a parallelogram into four equivalent parts by lines drawn from one of the vertices.

Ex. 814. Divide a parallelogram into two equivalent parts by a line parallel to a given line.

Ex. 815. On a given line as a base construct a rectangle equivalent to a given rhombus.

Ex. 816. On a given line as a base construct a rectangle equivalent to a given triangle.

Ex. 817. Given a square whose side is a, required to construct an equivalent isosceles triangle whose altitude is h.

Ex. 818. Construct a parallelogram having a given altitude and equivalent to a given trapezoid.

Ex. 819. Construct an isosceles triangle having a given base and equivalent to two thirds of a given trapezoid.

Ex. 820. Given a square whose side is b, required to construct an equivalent right triangle whose hypotenuse equals a given line k.

Ex. 821. Bisect a trapezoid by a line drawn from a given point in the shorter base.

Ex. 822. Construct a square equivalent to the sum of two given triangles.

Sug. First construct squares equivalent to the given triangles.

Ex. 823. Transform a given triangle into an isosceles triangle having a leg equal to a given line k.

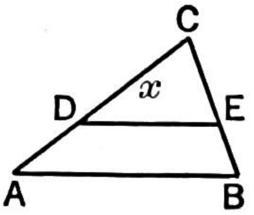
Ex. 824. Transform a given triangle into an isosceles triangle having the base equal to a given line k.

Ex. 825. Transform a square into a parallelogram having two of its sides equal respectively to two given lines.

Sug. Two transformations are necessary.

Ex. 826. Divide a triangle into two equivalent parts by a line drawn parallel to the base.

Analysis. $\frac{\triangle CDE}{\triangle CAB} = \frac{x^2}{\overline{CA}^2} = \frac{1}{2}$ ∴ etc.



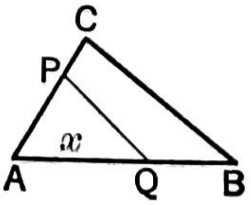
Ex. 827. Divide a quadrilateral into two equivalent parts.

Sug. Transform the quadrilateral into a triangle.

Ex. 828. Bisect the area of $\triangle ABC$ by a line drawn from a point P in AC.

Analysis. $\frac{\triangle APQ}{\triangle ABC} = \frac{AP\cdot x}{AC\cdot AB} = \frac{1}{2}$ $\therefore 2AP\cdot x = AC\cdot AB$

$\therefore 2\,AP : AC = AB : x$

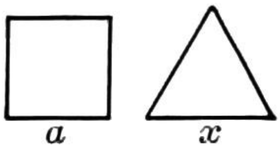


Ex. 829. Construct an equilateral triangle equivalent to a given square.

Analysis. $a^2 = \frac{1}{4}x^2\sqrt{3} \quad \therefore x^2 = \frac{4a^2}{\sqrt{3}}$

$$\therefore 2a : x = x : \frac{2a}{\sqrt{3}}$$

$$\therefore 2a : x = x : \tfrac{2}{3}a\sqrt{3}$$



Ex. 830. Divide a parallelogram into three equivalent parts by lines drawn from one of the vertices.

Ex. 831. Find a point within a triangle such that the lines joining the point to the vertices shall divide the triangle into three equivalent parts.

Ex. 832. Construct a polygon similar to two given similar polygons and equivalent to their sum.

Sug. Apply Ex. 730.

Ex. 833. Construct a polygon similar to two given similar polygons and equivalent to their difference.

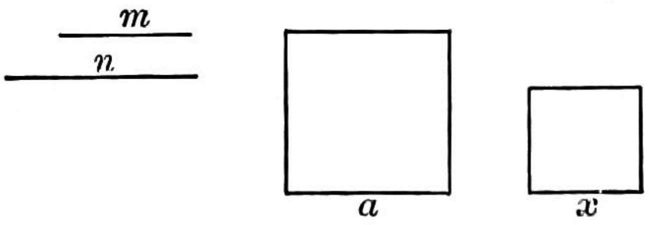
Ex. 834. Construct a square which shall be to a given square as $m : n$.

Analysis.

$x^2 : a^2 = m : n; \ \therefore x^2 = \frac{a^2m}{n}$

$\therefore a : x = x : \frac{am}{n}$

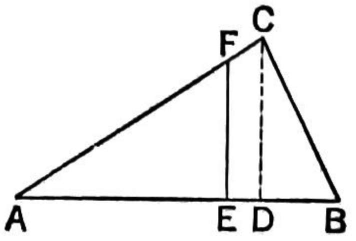
$\frac{am}{n}$ is readily found by getting the fourth proportional to n, m, and a.



Ex. 835. Divide a triangle into two equivalent parts by a line perpendicular to the base.

Analysis. $\frac{\triangle AEF}{\triangle ABC} = \frac{AE \cdot AF}{AB \cdot AC} = \frac{1}{2}$

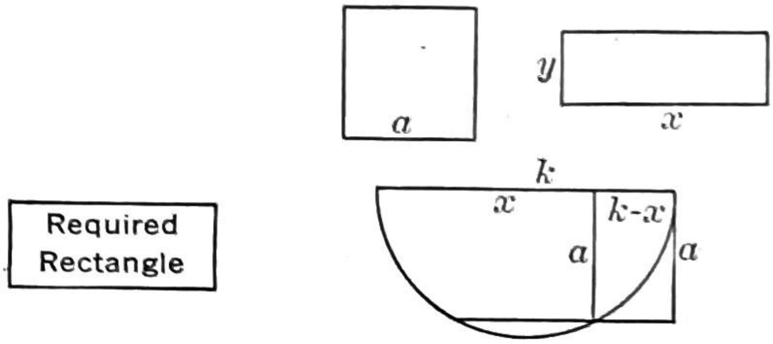
but $AE : AF = AD : AC; \ \therefore AF = \frac{AE \cdot AC}{AD}$



$\therefore \frac{AE}{AB\cdot \cancel{AC}}\cdot\frac{AE\cdot\cancel{AC}}{AD}=\frac{1}{2}$; $\therefore \overline{AE}^2=\frac{1}{2}AB\cdot AD$ $\therefore \frac{1}{2}AB : AE = AE : AD$.
Thus AE is determined.

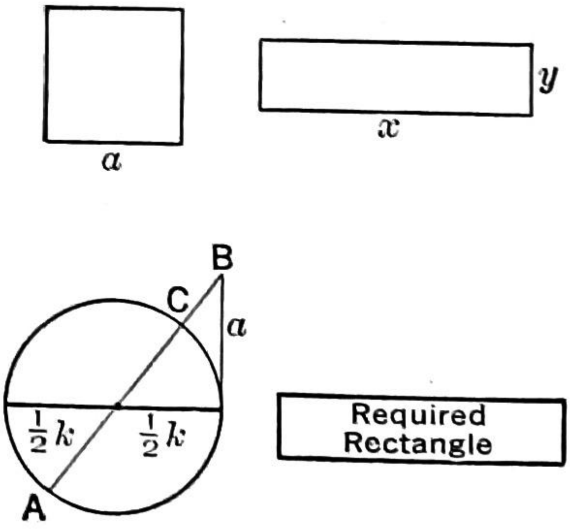
Ex. 836. Construct a rectangle equivalent to a given square, having the sum of its base and altitude equal to a given line.

Analysis.
$x + y = k;\ xy = a^2$
$\therefore x(k - x) = a^2$
$\therefore x : a = a : k - x$



Ex. 837. Construct a rectangle equivalent to a given square, having the difference of its base and altitude equal to a given line.

Analysis.
$x - y = k;\ xy = a^2$
$\therefore y = x - k$
$x(x - k) = a^2$
$\therefore x : a = a : x - k$
$AB = x,\ CB = x - k$
Hence construct a rectangle with base equal to AB and altitude equal to CB.



Ex. 838. Find a point P in $\triangle ABC$ such that the $\triangle$ PAB, PBC, and PCA are to each other as $3 : 4 : 5$.

Practical Applications

Ex. 839. A rectangular garden 60′ long and 40′ wide has a path 5′ wide around it. Find the area of the path.

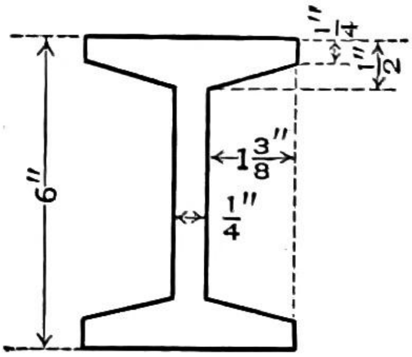
Ex. 840. At 18 cents per sq. ft., find the cost of building a cement walk 6′ wide on two sides of a corner lot 40′ by 100′.

Ex. 841. If sheet lead $\frac{1}{16}''$ thick weighs 3.69 lbs. to the square foot, how many pounds will be required to line the four sides and bottom of a tank 3′ 6″ deep, 4′ 8″ wide, and 8′ 4″ long?

Ex. 842. A steel bar can safely bear a pull or tension of 16,000 lbs. for each square inch of cross-section. How many pounds can a steel bar safely carry whose cross-section is a rectangle $2\frac{1}{2}''$ by $4\frac{1}{4}''$?

Ex. 843. A column carrying 320,000 lbs. rests on a square steel plate. What must be the size of the plate, assuming 16,000 lbs. per sq. in. as the safe load?

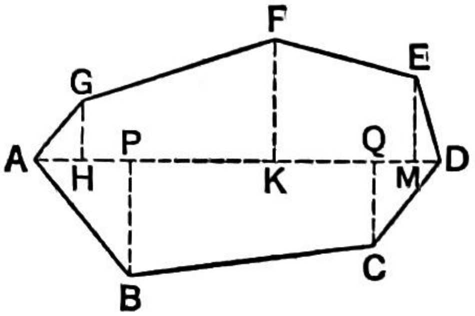
Ex. 844. Find the area of the cross-section of an I-beam with dimensions as shown in the figure.



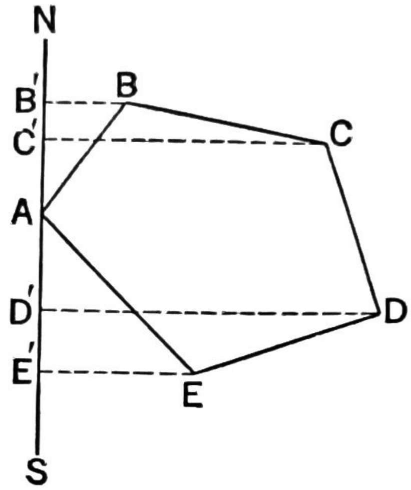
Ex. 845. Find the area of the cross-section of a railway embankment that is 32′ high, 75′ wide at the top, and 120′ wide at the bottom.

Ex. 846. A roof in the shape of a triangle is to be covered with patent roofing at 35 cents per sq. ft. If the sides are 80′, 100′, and 120′, find the cost of covering the roof.

Ex. 847. One way to find the area of a field in the shape of an irregular polygon is to divide the field into triangles and trapezoids and then sum their areas. A surveyor obtained the following measure-

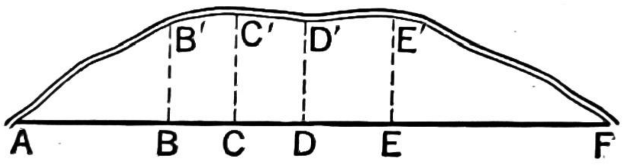


ments for the field here shown: $AH = 180'$, $HK = 590'$, $KM = 502'$, $MD = 95'$, $HP = 188'$, $QM = 165'$, $GH = 210'$, $FK = 385'$, $EM = 242'$, $BP = 383'$, $CQ = 247'$. Compute the area of the field.

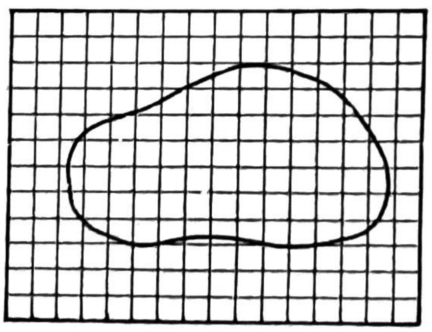


Ex. 848. To find the area of the field $ABCDE$ a surveyor laid off a north-and-south line NS through A, and found that $BB' = 150'$, $CC' = 456'$, $DD' = 490'$, $EE' = 210'$, $AB' = 190'$, $B'C' = 90'$, $C'D' = 238'$, $D'E' = 97'$. Compute the area of the field.

Ex. 849. The adjoining figure shows a field bounded by the shore of a lake. A close approximation to the area of this field is obtained by dividing it into triangles and trapezoids as here shown (by offsets run perpendicular to AF). We aim to get the curved boundaries of the figure as nearly straight as possible. Compute the area of this field from the following measurements: $AB = 210'$, $BC = 96'$, $CD = 95'$, $DE = 115'$, $EF = 240'$, $BB' = 142'$, $CC' = 150'$, $DD' = 138'$, $EE' = 140'$.



Ex. 850. Areas may be approximately found by the use of squared paper. The figure is usually drawn to some scale that uses the side of one of the small squares as a unit. The area is found by counting the number of squares within the boundary of the figure, considering as whole



squares those that are half or more than half within the boundary, and neglecting those that are less than half within the boundary. How many square feet are there in the figure here shown, if a side of one of the small squares equals 1 ft.?

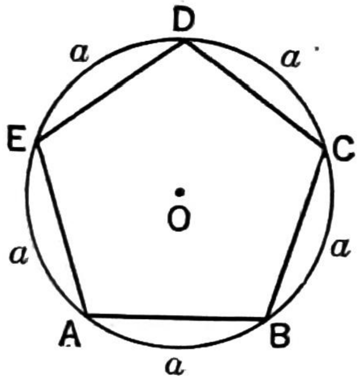
BOOK V

REGULAR POLYGONS AND THE CIRCLE

301. A *regular polygon* is one that has all its sides equal and all its angles equal. In other words, a regular polygon is both equilateral and equiangular.

Proposition I. Theorem

302. An equilateral polygon inscribed in a circle is a regular polygon.



Given equilateral polygon $ABCDE$ inscribed in $\odot\ O$.

Prove $ABCDE$ is a regular polygon.

Plan is to prove all the angles equal, by proving that they have the same measure.

Proof.

(1) $AB = BC = CD = DE = EA$	(1) Given.
(2) $\therefore \widehat{AB} = \widehat{BC} = \widehat{CD} = \widehat{DE} = \widehat{EA}$. We may therefore denote each of these arcs by the same symbol, such as a.	(2) 138.

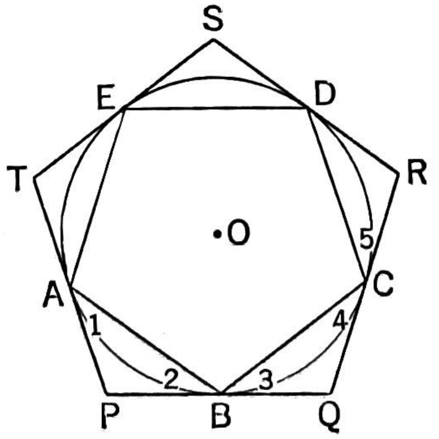
(3) $\angle A$ is measured by $\frac{1}{2}(3\ a)$	(3) 176.
(4) In like manner each angle is measured by $\frac{1}{2}(3\ a)$	(4) 176.
(5) $\therefore \angle A = \angle B = \angle C = \angle D = \angle E$	(5) Angles having same measure are equal.
(6) $\therefore$ polygon $ABCDE$ is regular	(6) 301.

Q.E.D.

INSCRIBED AND CIRCUMSCRIBED POLYGONS

Proposition II. Theorem

303. If a circle is divided into three or more equal arcs, the chords joining the successive points of division form a regular inscribed polygon; and the tangents at the points of division form a regular circumscribed polygon.



Given ⊙ O divided into equal arcs by points A, B, C, D, E, and these points joined in succession, thus forming the inscribed polygon $ABCDE$; and tangents drawn at A, B, C, D, E, thus forming the circumscribed polygon $PQRST$.

Prove $ABCDE$ and $PQRST$ are regular polygons.

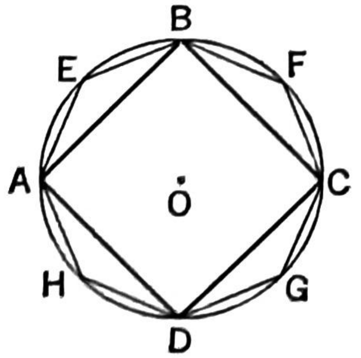
Plan for the first part is to use 302; *for the second part is to show that all the sides are equal and all the angles are equal.*

Proof.

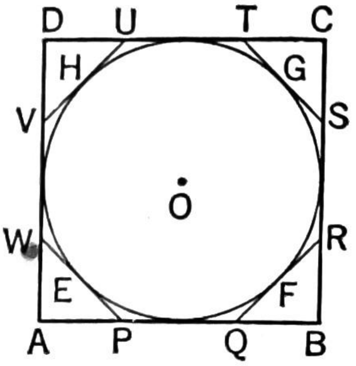
(1) $\overset{\frown}{AB} = \overset{\frown}{BC} = \overset{\frown}{CD} = \overset{\frown}{DE} = \overset{\frown}{EA}$	(1) Given.
(2) $\therefore AB = BC = CD = DE = EA$	(2) 139.
(3) $\therefore ABCDE$ is regular	(3) 302.
Second Part	
(1) $\angle 1$ is measured by $\frac{1}{2}\overset{\frown}{AB}$ $\angle 2$ " " " $\frac{1}{2}\overset{\frown}{AB}$ $\angle 3$ " " " $\frac{1}{2}\overset{\frown}{BC}$ $\angle 4$ " " " $\frac{1}{2}\overset{\frown}{BC}$ etc.	(1) 181.
(2) but $\overset{\frown}{AB} = \overset{\frown}{BC} = \overset{\frown}{CD} =$ etc.	(2) Given.
(3) $\therefore \angle 1 = \angle 2 = \angle 3 = \angle 4 =$ etc.	(3) Angles having equal measures are equal
(4) $AB = BC = CD =$ etc.	(4) 139.
(5) $\therefore \triangle APB \cong \triangle BQC \cong \triangle CRD$, etc.	(5) 38.
(6) $\therefore \angle P = \angle Q = \angle R =$ etc., and $PB = BQ = QC = CR =$ etc.	(6) Corr. parts of congruent ⚠ are equal.
(7) $\therefore PQ = QR = RS =$ etc.	(7) Ax. 2.
(8) $\therefore PQRST$ is regular.	(8) 301.

Q.E.D.

304. *Cor.* 1. If the midpoints of the arcs subtended by the sides of a regular inscribed polygon are joined to the vertices of the polygon, a regular inscribed polygon of double the number of sides is formed.

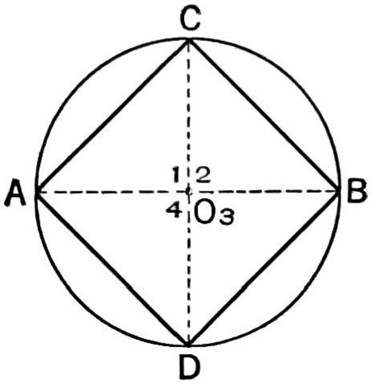


305. *Cor.* 2. If tangents are drawn at the midpoints of the arcs of adjacent points of contact of the sides of a regular circumscribed polygon, a regular circumscribed polygon of double the number of sides is formed.



Proposition III. Problem

306. To inscribe a square in a given circle.



Given ⊙ *O*.

Required to inscribe a square in ⊙ *O*.

Plan is to use 303.

Construction. (*a*) Draw any two diameters perpendicular to each other, such as *AB* and *CD*.

(*b*) Draw *AC*, *CB*, *BD*, *DA*. Then *ADBC* is the required square.

Proof.

(1) $\angle 1 = \angle 2 = \angle 3 = \angle 4$	(1) All right angles are equal.
(2) $\therefore \overset{\frown}{AC} = \overset{\frown}{CB} = \overset{\frown}{BD} = \overset{\frown}{DA}$	(2) 133.
(3) $\therefore$ *ADBC* is a regular polygon, i.e. a square.	(3) 303.

Q.E.D.

307. *Cor.* 1. By successive applications of 304, regular polygons of 8, 16, 32, 64, etc., sides may be inscribed in a circle.

Ex. 851. A square is inscribed in a circle of 4″ radius. Find a side of the square and the area of the square.

Ex. 852. The side of a square inscribed in a given circle is 8″. Find the radius of the circle.

Ex. 853. Circumscribe a square about a given circle.

Ex. 854. A square is circumscribed about a circle of 5″ radius. Find the area of the square.

Ex. 855. If the radius of a circle is r, find the length of a side of the inscribed square; of the circumscribed square.

Ex. 856. Circumscribe a regular octagon about a given circle.

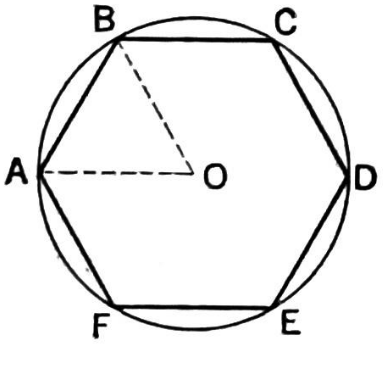
Ex. 857. A regular octagon is inscribed in a circle of 10″ radius. Find to the nearest tenth of an inch the length of a side.

Ex. 858. A regular octagon is circumscribed about a circle of 10″ radius. Find to the nearest tenth of an inch the length of a side.

Ex. 859. Prove that an equiangular polygon circumscribed about a circle is regular.

Proposition IV. Problem

308. To inscribe a regular hexagon in a given circle.



Given $\odot\ O$.

Required to inscribe a regular hexagon in $\odot\ O$.

Plan is to divide the circle into six equal arcs, and then use 303.

Construction. (*a*) Draw any radius OA.

(*b*) With A as center, and AO as radius, draw an arc cutting the circle at B.

(*c*) With B as center, and the same radius, draw an arc cutting the circle at C.

(*d*) Continue this process, and thus obtain the points D, E, F.

(*e*) Join these points in succession. Then $ABCDEF$ is the required regular hexagon.

Proof. Draw OB.

(1) Then $\triangle\ AOB$ is equilateral Why?

(2) $\therefore\ \angle\ AOB = 60°$ Why?

(3) $\therefore\ \widehat{AB} = 60° = \frac{1}{6}$ of $360° = \frac{1}{6}$ of $\odot\ O$ Why?

In like manner each of the arcs BC, CD, DE, EF may be proved equal to $\frac{1}{6}$ of the $\odot\ O$.

(4) $\therefore\ \widehat{FA} = \frac{1}{6}$ of $\odot\ O$ Why?

(5) $\therefore\ ABCDEF$ is a regular hexagon. Why?

309. *Cor.* 1. If the alternate vertices of a regular inscribed hexagon are joined, an inscribed equilateral triangle is formed.

310. *Cor.* 2. By successive applications of 304, regular polygons of 12, 24, 48, 96, ... sides may be inscribed in a circle.

311. In the appendix it is shown how to inscribe a regular polygon of 5 sides, of 10 sides, and of 15 sides in a circle. Thus we see that by the use of ruler and compasses the following regular polygons can be inscribed in a circle:

3, 6, 12, 24, 48, ... 3×2^n sides
4, 8, 16, 32, 64, ... 4×2^n sides
5, 10, 20, 40, 80, ... 5×2^n sides
15, 30, 60, 120, 240, ... 15×2^n sides

For a long time it was thought that these were the only regular polygons that could be constructed with ruler and compasses until, in 1796, Gauss, a brilliant German mathematician, showed how a regular polygon of 17 sides could be constructed. The construction and proof are beyond the scope of this book. He proved the general theorem that regular polygons with a prime number of sides can be inscribed in a circle by means of ruler and compasses if and only if the prime number is of the form $2^n + 1$, where n is a positive integer. Though regular polygons of 7, 9, 11, 13, 14, 18, 19 sides cannot be geometrically constructed with ruler and compasses, there are approximate ways of constructing them.

Ex. 860. Circumscribe a regular hexagon about a given circle.

Ex. 861. Circumscribe an equilateral triangle about a given circle.

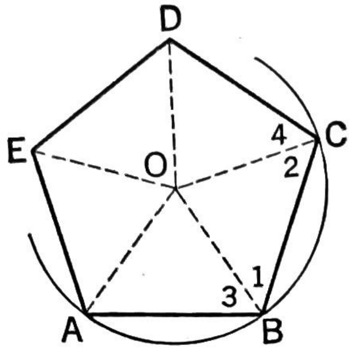
Ex. 862. The radius of a circle is 3″. Find a side of the inscribed equilateral triangle; of the circumscribed equilateral triangle.

Ex. 863. Find the ratio of the side of a square inscribed in a circle to the side of an equilateral triangle inscribed in the same circle.

Ex. 864. In any circle find the ratio of the area of the inscribed square to that of the circumscribed square; of the inscribed equilateral triangle to that of the circumscribed equilateral triangle; of the inscribed regular hexagon to that of the circumscribed regular hexagon.

Proposition V. Theorem

312. A circle may be circumscribed about any regular polygon.



Given regular polygon *ABCDE.*

Prove a circle may be circumscribed about *ABCDE.*

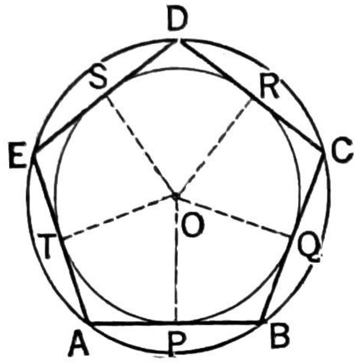
Plan is to show that the circle which we know can be made to pass through A, B, C (see 154) *must also pass through D and E.*

Proof.

(1) Pass a circle through A, B, C, and let O be its center. Draw the radii OA, OB, OC, and also draw lines OD and OE.	(1) 154.
(2) $OB = OC$	(2) Why?
(3) $\therefore \angle 1 = \angle 2$	(3) Why?
(4) but $\angle ABC = \angle BCD$	(4) 301.
(5) $\therefore \angle 3 = \angle 4$	(5) Ax. 3.
(6) $AB = CD$	(6) 301.
(7) $\therefore \triangle OAB \cong \triangle OCD$	(7) 40.
(8) $\therefore OA = OD$, i.e. the ⊙ will also pass through D.	(8) Why?
(9) In like manner it can be shown that the ⊙ will pass through E. Hence the ⊙ constructed with O as center and OA as radius is circumscribed about $ABCDE$.	(9) 151.
	Q.E.D.

Proposition VI. Theorem

313. A circle may be inscribed in any regular polygon.



Given regular polygon $ABCDE$.

Prove a circle may be inscribed in $ABCDE$.

Plan is to locate a point that is equidistant from the sides of ABCDE.

Proof.

(1) Circumscribe a ⊙ about $ABCDE$, and let O be the center of this ⊙. Draw OP, OQ, OR, OS, OT perpendicular respectively to AB, BC, CD, DE, EA	(1) 312.
(2) Then $OP = OQ = OR = OS = OT$	(2) 144.
(3) Hence the ⊙ drawn with O as center and OP as radius will pass through P, Q, R, S, T	(3) $OP = OQ = OR$ $= OS = OT$
(4) ∴ the sides of $ABCDE$ will be tangent to this ⊙	(4) 159.
(5) ∴ the ⊙ will be inscribed in $ABCDE$	(5) 163.

Q.E.D.

314. The *center of a regular polygon* is the center of both the inscribed and circumscribed circles. Thus O is the center of $ABCDE$.

315. The *radius* of a regular polygon is the radius of the circumscribed circle. Thus OA is the radius of $ABCDE$.

316. The *apothem* of a regular polygon is the radius of the inscribed circle. Thus OP is the apothem of $ABCDE$.

317. The *angle at the center* of a regular polygon is the angle formed by two radii drawn to the extremities of any side. Thus $\angle AOB$ is the angle at the center of $ABCDE$. In a regular n-gon the central angle is $\frac{360°}{n}$.

Exercises

Ex. 865. The radius of a regular polygon bisects the angle at the vertex to which it is drawn.

Ex. 866. On a given line as a side, construct a regular hexagon.

Ex. 867. On a given line as a side, construct a regular octagon.

Ex. 868. A square and a regular octagon are inscribed in a circle. Prove that a side of the octagon is less than a side of the square, and that the perimeter of the octagon is greater than the perimeter of the square.

Ex. 869. A side of an inscribed equilateral triangle bisects the radius drawn to the midpoint of the subtended arc.

Ex. 870. The apothem of a regular polygon bisects the side to which it is drawn.

Ex. 871. A principal diagonal of a regular hexagon passes through the center of the circumscribed circle, and is parallel to a pair of opposite sides.

Ex. 872. The opposite sides of a regular octagon are parallel.

Ex. 873. The diagonals of a regular pentagon are equal.

Ex. 874. The diagonals from the vertex of a regular n-gon divide the angle at the vertex into $(n - 2)$ equal parts.

Ex. 875. If two diagonals of a regular pentagon intersect, the greater segment of each is equal to a side of the pentagon.

Ex. 876. Prove that any angle of a regular polygon is the supplement of the angle at the center, and hence that an exterior angle of a regular polygon is equal to the angle at the center.

Ex. 877. A polygon is regular if its inscribed and circumscribed circles are concentric.

Ex. 878. If squares are constructed outwardly on the sides of a regular hexagon, the outer vertices of the squares are the vertices of a regular polygon of 12 sides.

Ex. 879. Tangents to a circle at the midpoints of the arcs of the sides of a regular inscribed polygon form a regular circumscribed polygon whose sides are parallel to the sides of the inscribed polygon and whose vertices lie on radii produced of the inscribed polygon.

GROUP XVI

State whether the following statements are true or false, and give reasons.

(*a*) The radii of a regular polygon divide the polygon into congruent triangles.

(*b*) A central angle of a regular polygon cannot exceed 120°.

(*c*) The apothem of a regular polygon is less than the radius of the polygon.

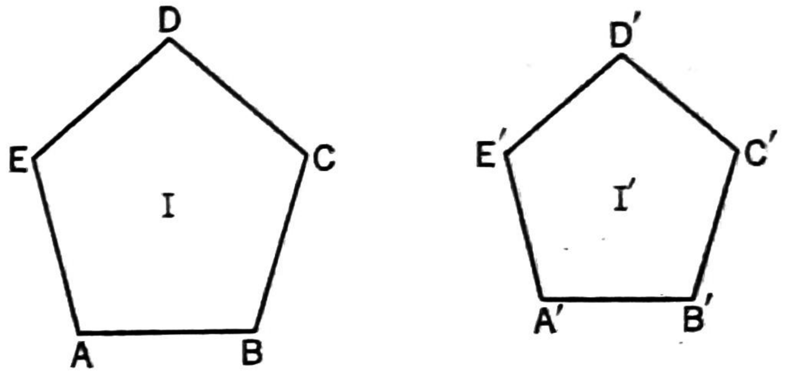
(*d*) If the number of sides of a regular inscribed polygon is doubled, the perimeter is doubled.

(*e*) If a square and a regular octagon are circumscribed about a circle, the perimeter of the square exceeds that of the octagon.

SIMILAR REGULAR POLYGONS

Proposition VII. Theorem

318. Two regular polygons of the same number of sides are similar.



Given two regular polygons I and I', each having n sides

Prove $I \sim I'$.

Plan is to show that the two conditions for similarity are here fulfilled.

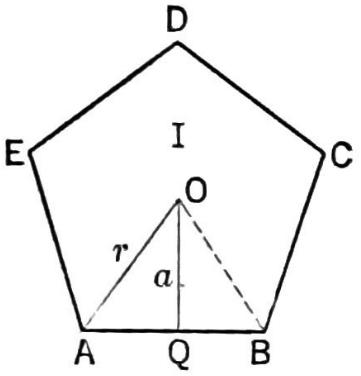
Proof.

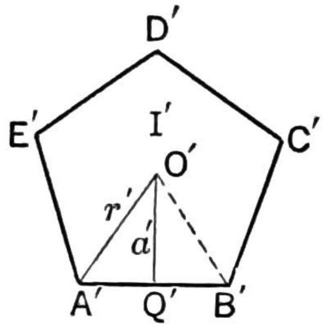
(1) Each $\angle$ of $I = \dfrac{(n-2)\,180^\circ}{n}$	(1) 113.
Each $\angle$ of $I' = \dfrac{(n-2)\,180^\circ}{n}$	
(2) $\therefore$ I and I' are mutually equiangular	(2) Ax. 1,
(3) Also $AB = BC = CD =$ etc.	(3) 301.
and $A'B' = B'C' = C'D' =$ etc.	
(4) $\therefore \dfrac{AB}{A'B'} = \dfrac{BC}{B'C'} = \dfrac{CD}{C'D'} =$ etc.	(4) Ax. 5.
(5) $\therefore I \sim I'$	(5) 231.
	Q.E.D.

Ex. 880. The corresponding sides of two regular octagons are 6″ and 8″ respectively. Find the ratio of their perimeters; also of their areas.

Proposition VIII. Theorem

319. The perimeters of two regular polygons of the same number of sides are to each other as their radii and also as their apothems.





Given two regular polygons I and I', each having n sides, with perimeters p and p', radii r and r', and apothems a and a' respectively.

Prove $\frac{p}{p'} = \frac{r}{r'} = \frac{a}{a'}$.

Analysis. We know that $\frac{p}{p'} = \frac{AB}{A'B'}$. *Hence it only remains to show that* $\frac{AB}{A'B'} = \frac{r}{r'} = \frac{a}{a'}$. *This readily follows from the similarity of* $\triangle$s *AOB and A′O′B′.*

Proof. Draw the radii OB and $O'B'$.

(1) $\angle AOB = \angle A'O'B'$	(1) Each is equal to $\frac{360°}{n}$.
(2) $\frac{OA}{OB} = \frac{O'A'}{O'B'}$	(2) Each ratio equals 1.
(3) $\therefore \triangle OAB \sim \triangle O'A'B'$	(3) 244.
(4) $\therefore \frac{AB}{A'B'} = \frac{r}{r'}$	(4) Corr. sides of $\sim$ $\triangle$s are in proportion.
(5) $\angle AOQ = \angle A'O'Q'$	(5) Each is equal to $\frac{180°}{n}$.
(6) $\therefore \triangle OAQ \sim \triangle O'A'Q'$	(6) 238.

(7) $\therefore \frac{r}{r'} = \frac{a}{a'}$	(7) Corr. sides of $\sim$ △ are in proportion.
(8) but $\frac{p}{p'} = \frac{AB}{A'B'}$	(8) 259.
(9) $\therefore \frac{p}{p'} = \frac{r}{r'} = \frac{a}{a'}$	(9) Ax. 1.

Q.E.D.

320. *Cor.* The areas of two regular polygons of the same number of sides are to each other as the squares of their radii and also as the squares of their apothems.

Ex. 881. The perimeters of two regular pentagons are 25″ and 30″ respectively. Find the ratio of their radii; of their apothems; of their areas; of any two corresponding sides; of any two corresponding diagonals.

GROUP XVII

Complete each of the following statements:

(*a*) All regular hexagons are . . . .

(*b*) A circle can always be circumscribed about a polygon if the polygon is . . . .

(*c*) The midpoints of the sides of a regular polygon lie in a circle whose radius is called the . . . of the regular polygon.

(*d*) If the area of a square inscribed in a circle is 9 sq. in., then the area of a square circumscribed about the same circle is . . . .

(*e*) The areas of two regular octagons are in the ratio 9 : 49. If the perimeter of the larger octagon is 48, the perimeter of the smaller is . . . . The ratio of their radii is . . . .

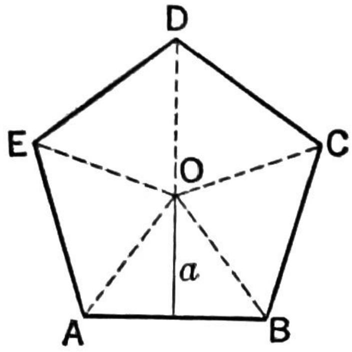
The ratio of their apothems is . . . .

(*f*) If a regular pentagon is inscribed in a circle, a tangent at one of its vertices makes an angle of . . . degrees with one of the sides of the pentagon.

AREA OF A REGULAR POLYGON

Proposition IX. Theorem

321. The area of a regular polygon is equal to one half the product of its apothem and its perimeter.



Given regular polygon $ABCDE$ with perimeter p, apothem a, and area K.

Prove $K = \frac{1}{2} a \times p$.

Plan is to divide the polygon into triangles, and then sum their areas.

Proof. Draw the radii to all the vertices. Then

(1) Area $\triangle OAB = \frac{1}{2} AB \times a$ Area $\triangle OBC = \frac{1}{2} BC \times a$ Area $\triangle OCD = \frac{1}{2} CD \times a$ ⋮ ⋮	(1) 284.
(2) $\therefore \triangle OAB + \triangle OBC + \triangle OCD +$ etc. $= \frac{1}{2} AB \times a + \frac{1}{2} BC \times a + \frac{1}{2} CD \times a +$ etc.	(2) Ax. 2.
(3) $\therefore K = \frac{1}{2} a(AB + BC + CD + \text{etc.})$	(3) Ax. 6, and factoring.
(4) $\therefore K = \frac{1}{2} a \times p$	(4) Sub.

Q.E.D.

322. The above proposition might have been proved as follows: Let the polygon have n sides.

$$\text{Area } \triangle OAB = \tfrac{1}{2} a \times AB$$

$$\therefore n \times \text{area } \triangle OAB = \tfrac{1}{2} a \times AB \times n$$

$$\therefore K = \tfrac{1}{2} a \times p.$$

Ex. 882. Find the area of a regular hexagon whose side is 8″.

Ex. 883. The apothem of a regular hexagon is $2\sqrt{3}''$. Find the area of the hexagon.

Ex. 884. If an equilateral triangle and a regular hexagon are inscribed in the same circle, the area of the triangle is half the area of the hexagon.

Ex. 885. The area of a regular hexagon inscribed in a circle is a mean proportional between the areas of the inscribed and circumscribed equilateral triangles.

Ex. 886. Find the area of a regular octagon whose side is 4″.

Circle Measurement

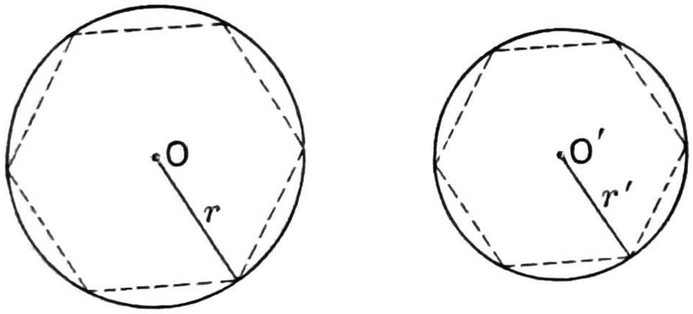
323. We define the length of a straight line as the number of times a certain unit straight line goes into the given line. Obviously this definition can not apply to the length of a circle. But we may obtain the approximate length of a circle as follows: Imagine a square inscribed in a circle, then form a regular inscribed polygon of 8 sides, then one of 16 sides, 32 sides, 64 sides, and keep on doubling the number of sides. The further we continue this process, the closer will the perimeter of the polygon approach the circumference of the circle; the area of the polygon will approach the area inclosed by the circle; and the apothem of the polygon will approach the radius of the circle. If we were to circumscribe a square about a circle, and repeatedly double the number of sides, the perimeter of the polygon will get closer and closer to the circumference of the circle; the area of the polygon will get closer and closer to the area inclosed by the circle; but the apothem of the polygon is always equal to the radius of the circle. Hence by calculating the lengths of the perimeters of the inscribed and circumscribed polygons, we approximate closer and closer to the length of the circle.

324. We shall assume then that *from a study of the properties of regular polygons we may arrive at the corresponding properties of circles.* The proof of this statement is based on the theory of limits.

CIRCUMFERENCES

Proposition X. Theorem

325. The circumferences of two circles are to each other as their radii.



Given circles O and O' with circumferences c and c', and radii r and r' respectively.

Prove $\frac{c}{c'} = \frac{r}{r'}$.

Plan is to inscribe regular polygons of the same number of sides, and obtain the ratio of their perimeters as the number of sides is repeatedly doubled.

Proof. Inscribe regular polygons of the same number of sides, and denote their perimeters by p and p' respectively.

Then

(1) $\frac{p}{p'} = \frac{r}{r'}$	(1) 319.
(2) If we repeatedly double the number of sides of these polygons, the ratio $\frac{p}{p'}$ will get closer and closer to the ratio $\frac{c}{c'}$.	(2) 324.
$\therefore \frac{c}{c'} = \frac{r}{r'}$	Q.E.D.

326. *Cor.* 1. The circumferences of two circles are to each other as their diameters. For $\frac{c}{c'} = \frac{r}{r'} = \frac{2\,r}{2\,r'} = \frac{d}{d'}$.

327. *Cor.* 2. The ratio of the circumference of any circle to its diameter is constant (i.e. the ratio is the same for any two circles). For $\frac{c}{c'} = \frac{d}{d'}$ $\therefore \frac{c}{d} = \frac{c'}{d'}$ (by alternation).

328. *Cor.* 3. This constant ratio is represented by the Greek letter π (pi). $\therefore \frac{c}{d} = \pi$. Hence $c = \pi d = 2\,\pi r$.

329. How to obtain the value of π is shown in 354. Its value is approximately $3\frac{1}{7}$; a closer approximation is 3.1416.

330. Since $\frac{\text{arc}}{\text{circumference}} = \frac{\text{central angle subtended by arc}}{360°}$ we can compute the length of an arc of a circle if we know the number of degrees in the arc and also the length of the radius. For example: In a circle of 10″ radius find the length of an arc of 20°.

Solution. arc $= \frac{20}{360}$ of circumference
$= \frac{1}{18}$ of $2 \times 3.1416 \times 10$
$= 3.491$
$= 3.5''$ (to the nearest tenth of an inch)

Ex. 887. Find the circumference of a circle whose radius is 28″. (Use $\pi = 3\frac{1}{7}$)

Ex. 888. Find the circumference of a circle whose diameter is 3.066 ft. (Use $\pi = 3\frac{1}{7}$)

Ex. 889. Find the radius of a circle whose circumference is 48″. (Use $\pi = 3\frac{1}{7}$)

Ex. 890. Find the diameter of a circle whose circumference is 56.1″. (Use $\pi = 3\frac{1}{7}$)

Ex. 891. The radii of two circles are 5′ and 6′ respectively. Find the ratio of their diameters; of their circumferences.

Ex. 892. In a circle whose diameter is 8″ find the length of an arc of 60°; of $22\frac{1}{2}°$; of 8°.

Ex. 893. How many degrees are there in an arc 20 ft. long, if the radius of the circle is 15 ft.?

Ex. 894. A wheel is 3′ 6″ in diameter. How many revolutions will it make in going half a mile?

Ex. 895. What is the diameter of a wheel that makes 300 revolutions in traveling one mile?

Ex. 896. Draw a circle whose circumference shall equal the sum of the circumferences of two given circles.

Ex. 897. Draw a circle whose circumference shall equal the difference between the circumferences of two given circles.

Ex. 898. The circumferences of two concentric circles differ by 8″. Compute the width of the ring between the two circles.

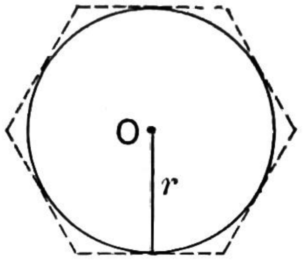
Ex. 899. An arc of a circle is equal in length to a radius of the circle. How many degrees are there in the arc?

331. The method of 271 cannot be used to find the area inclosed by a circle, but 323 suggests a way of approximating to this area. It should be noted that a circle is a *line*, and hence to speak of "the area of a circle" is nonsense. However, the expression "area of a circle" is commonly used, and we shall use it in this text with the understanding that it is an abbreviation for the expression "area inclosed by a circle."

AREA OF CIRCLE

Proposition XI. Theorem

332. **The area of a circle is equal to one half the product of its circumference and its radius.**



Given ⊙ O, with its radius r, circumference c, and area K.

Prove $K = \frac{1}{2} c \times r$.

Plan is to circumscribe a regular polygon about the ⊙, *and obtain an expression for its area as the number of sides is repeatedly doubled.*

Proof. Circumscribe a regular polygon about the circle, and denote its area by K', its perimeter by p, and its apothem by r. Then

(1) $K' = \frac{1}{2} p \times r$ — (1) 321.

(2) If we repeatedly double the number of sides of this circumscribed polygon, K' will get closer and closer to K, p will get closer and closer to c, but r is always the same. Hence — (2) 324.

$K = \frac{1}{2} c \times r$ — Q.E.D.

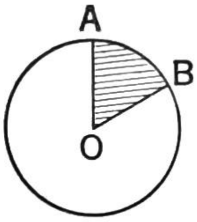
333. *Cor.* 1. **The area of a circle equals πr^2 or $\frac{1}{4} \pi d^2$.**

For $K = \frac{1}{2} c \times r = \frac{1}{2} (2 \pi r) r = \pi r^2 = \pi \left(\frac{d}{2}\right)^2 = \frac{1}{4} \pi d^2$.

334. *Cor.* 2. **The areas of two circles are to each other as the squares of their radii, or as the squares of their diameters.**

For $\frac{K_1}{K_2} = \frac{\pi r_1^2}{\pi r_2^2} = \frac{r_1^2}{r_2^2} = \frac{(\frac{1}{2} d_1)^2}{(\frac{1}{2} d_2)^2} = \frac{d_1^2}{d_2^2}$.

Sectors and Segments



335. A *sector* of a circle is the area inclosed by two radii and the arc intercepted by them. Thus AOB is a sector of ⊙ O.

336. Since $\frac{\text{area of sector}}{\text{area of circle}} = \frac{\text{angle of sector}}{360°}$ we can compute the area of a sector if we know the radius of the circle and the number of degrees in the arc of the sector. For example: In a circle of 5″ radius find the area of a sector of 40°.

Solution. Area of sector $= \frac{40}{360}$ of area of ⊙.
$= \frac{1}{9}$ of 3.1416×5^2
$= \frac{1}{9}$ of 78.54
$= 8.73$ sq. in. (to the nearest hundredth)

337. *Cor.* 3. The area of a sector of a circle is equal to one half the product of the radius and the arc of the sector.

For
$$\frac{\text{area } AOB}{\text{area } \odot} = \frac{\angle AOB}{360°} = \frac{\widehat{AB}}{\text{circumference}}$$

$$\therefore \text{area } AOB = \frac{\widehat{AB}}{\text{circumference}} \times \text{area of } \odot.$$
$$= \frac{\pi r^2}{2\pi r} \times \widehat{AB}$$
$$= \tfrac{1}{2} r \times \widehat{AB}.$$

338. The area of a segment of a circle is found by subtracting the area of the triangle (formed by the radii and the chord of the segment) from the area of the sector.

339. In circles of unequal radii, *similar sectors* are sectors that correspond to equal central angles.

340. In circles of unequal radii, *similar segments* are segments that correspond to equal central angles.

341. *Cor.* 4. Similar sectors are to each other as the squares of their radii.

$$\text{For } \frac{S_1}{S_2} = \frac{\frac{A}{360}\pi r_1^2}{\frac{A}{360}\pi r_2^2} = \frac{r_1^2}{r_2^2}.$$

Exercises

Ex. 900. Find the area of a circle whose diameter is 9″.

Ex. 901. Find the area of a circle whose circumference is 32 π″.

Ex. 902. Find the ratio of the areas of two circles whose diameters are 8″ and 10″ respectively; also find the ratio of their circumferences.

Ex. 903. If the radius of a circle is doubled, what is the effect upon the circumference? upon the area?

Ex. 904. If the area of a circle is doubled, what is the effect upon the radius? upon the circumference?

Ex. 905. Construct a circle equivalent to the sum of the areas of two given circles.

Ex. 906. Construct a circle equivalent to the difference of the areas between two given circles.

Ex. 907. Construct a circle whose area shall be five times the area of a given circle.

Ex. 908. The areas of two circles are to each other as 9 : 1. Find the ratio of their circumferences.

Ex. 909. Find the radius of a circle whose area is 100 sq. in.

Ex. 910. In a circle of diameter 8″, find the area of a sector of 36°.

Ex. 911. In a circle of 14″ radius, find the area of a sector whose arc is 10″ long. How many degrees are there in the arc?

Ex. 912. A parallelogram with adjacent sides 5″ and 12″ is inscribed in a circle. Find the area of the circle.

Ex. 913. Prove that the area of a circle is equal to four times the area of the circle described upon its radius as a diameter.

Ex. 914. Prove that the area of the ring included between two concentric circles is equal to the area of a circle whose diameter is that chord of the outer circle which is tangent to the inner circle.

Ex. 915. A circular grass plot 50 yds. in diameter is surrounded by a walk 2 yds. wide. Find the area of the walk.

Ex. 916. What is the area of the circular ring between the circles inscribed in and circumscribed about a square whose side is 15″?

Ex. 917. A regular hexagon is inscribed in a circle of 3″ radius. Find the area included between the hexagon and the circle.

Ex. 918. Two tangents to a circle from the same point include an angle of 60°. If the radius is 8″, find the area bounded by the two radii to the points of contact and the minor arc determined by the tangents.

Ex. 919. The area of a regular hexagon is 54 sq. in. Find the area of the circumscribed circle; of the inscribed circle.

Ex. 920. Two tangents to a circle of radius 10″ from an external point form an angle of 60°. Find the area of the segment bounded by the minor arc and the chord of contact.

Ex. 921. An equilateral triangle inscribed in a circle has an area equal to $144\sqrt{3}$ sq. in. What is the area of the circle?

Ex. 922. The area of a square is 72 sq. in. Find the area of the circumscribed circle; of the inscribed circle.

Ex. 923. The area of an equilateral triangle is $36\sqrt{3}$ sq. ft. Find the area of the inscribed circle; of the circumscribed circle.

Ex. 924. The radius of a circle is 18″. Find the area and the perimeter of a segment of this circle whose chord equals the radius.

Ex. 925. An equilateral triangle whose side is 6″ has the same area as a circle. Find the radius of the circle.

Ex. 926. A 6″ pipe supplies only three quarters as much water in a given time as is needed. Assuming that the flow is proportional to the area of the cross-section, find the diameter of the smallest pipe that will supply the required amount if pipe is made in diameters of exact number of inches.

Ex. 927. The perimeter of a square equals the circumference of a circle. Find the ratio of the areas of the two figures.

Ex. 928. The arc of a segment of a circle is 45°. If the radius of the circle is 20″, find the area of the segment.

Miscellaneous Theorems

Ex. 929. The bisectors of the angles of a polygon circumscribed about a circle meet in a point.

Ex. 930. A square constructed on a diameter of a circle is equivalent to twice the area of the inscribed square.

Ex. 931. Find the locus of the vertices of all regular polygons of the same number of sides that can be circumscribed about a given circle.

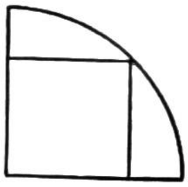
Ex. 932. Find the locus of the midpoints of the sides of all regular polygons of the same number of sides that can be inscribed in a given circle.

Ex. 933. *ABCDE* is a regular pentagon whose diagonals *AC* and *BE* intersect in *P*. Prove that *CDEP* is a parallelogram.

Ex. 934. The lines joining in succession the feet of the apothems of a regular polygon inscribed in a circle form a regular polygon of the same number of sides.

Ex. 935. The radius of an inscribed regular polygon is the mean proportional between its apothem and the radius of the similar circumscribed regular polygon.

Ex. 936. The apothem of an inscribed equilateral triangle is equal to half the radius.



Ex. 937. The area of the square inscribed in a sector whose central angle is a right angle is equal to half the square of the radius.

Ex. 938. Prove that the perimeter of a regular n-gon is less than the perimeter of the regular $2n$-gon, both being inscribed in the same circle.

Ex. 939. Prove that the perimeter of a regular n-gon is greater than the perimeter of the regular $2n$-gon, both being circumscribed about the same circle.

Ex. 940. The diagonals joining the alternate vertices of a regular hexagon form another regular hexagon.

Ex. 941. The diagonals of a regular pentagon form another regular pentagon.

Ex. 942. The lines joining in succession the midpoints of the radii of any regular polygon form a regular polygon of the same number of sides whose area is one fourth of the given polygon.

Ex. 943. The square of a side of an inscribed equilateral triangle is equal to the sum of the squares of a side of the square and of the regular hexagon inscribed in the same circle.

Ex. 944. An inscribed regular octagon is equivalent to a rectangle whose sides are equal to the sides of an inscribed and circumscribed square.

Ex. 945. The square inscribed in a semicircle is two fifths of the square inscribed in the whole circle.

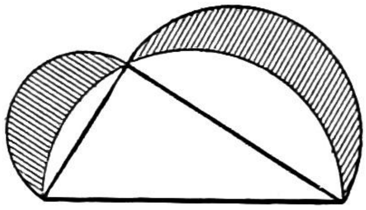
Ex. 946. In two circles of different radii, angles at the center subtended by arcs of equal length are to each other inversely as their radii.

Ex. 947. The square of a side of an inscribed equilateral triangle is equal to three times the square of a side of the inscribed regular hexagon.

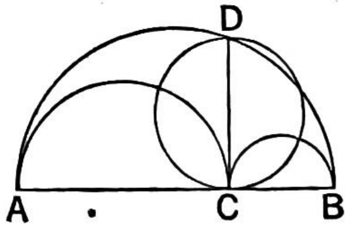
Ex. 948. The areas of two similar segments are to each other as the squares of their radii.

Sug. Prove $\text{sector}_1 : \text{sector}_2 = \triangle_1 : \triangle_2$, and then apply alternation and subtraction.

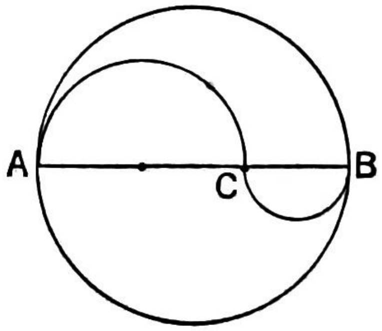
Ex. 949. Semicircles are constructed on the sides of a right triangle, as here shown. Prove that the sum of the areas of the two crescents is equal to the area of the triangle. (Note that we here have an instance of the area of a curvilinear figure equal to the area of a rectilinear figure.)



Ex. 950. In the annexed drawing, C is any point in line AB, and semicircles are drawn upon AB, AC, and CB as diameters. $CD \perp AB$ and terminates in the largest semicircle. Prove that these semicircles bound an area equivalent to a circle whose diameter is CD.



Ex. 951. If the diameter of a circle is divided into two parts, and upon these parts semicircles are constructed on opposite sides of the diameter, these semicircles will divide the area of the circle into two parts which have the same ratio as the segments of the diameter.



Ex. 952. An equiangular polygon inscribed in a circle is regular if the number of sides is odd.

Ex. 953. An equilateral polygon circumscribed about a circle is regular if the number of sides is odd.

Ex. 954. The areas of triangles inscribed in equal circles are to each other as the products of their three sides.

Sug. Use 270.

Ex. 955. If from any point within a regular *n*-gon perpendiculars are drawn to the sides of the polygon, the sum of these perpendiculars is equal to *n* times the apothem.

Ex. 956. If in a circle two chords are perpendicular to each other, and circles are constructed on the segments of the chords as diameters, the area of the given circle is equivalent to the sum of the areas of the four circles.

Sug. See Ex. 586.

Numerical Exercises

Ex. 957. If the diameter of a circle is 9″, find the diameter of a circle one fourth as large.

Ex. 958. If the radius of a circle is 21″, what is the radius of the concentric circle which divides the given circle into two equivalent parts?

Ex. 959. A circle has an area of 40 sq. in. Find the length of an arc of 40°.

Ex. 960. In a circle of 18″ radius the area of a sector is 36π sq. in. Find the number of degrees in the arc of the sector.

Ex. 961. The altitude of an equilateral triangle is 10″. Find the area of the inscribed circle; of the circumscribed circle.

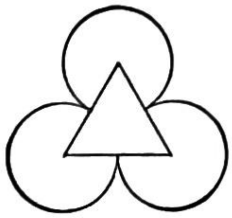
Ex. 962. The area of the circle circumscribed about an equilateral triangle is 9π sq. in. Find the altitude of the triangle.

Ex. 963. What is the area of a circle whose arc of $22\frac{1}{2}°$ is $\frac{3}{2}\pi$ ft.?

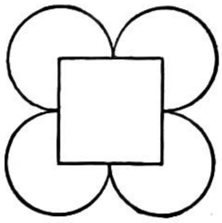
Ex. 964. Find the radius of a circle if its area is doubled by increasing the radius one foot.

Ex. 965. It requires $41\frac{1}{7}$ rods of fence to inclose a semicircular field. Find the area of the field. (Use $\pi = 3\frac{1}{7}$.)

Ex. 966. The trefoil here shown is based upon an equilateral triangle whose side is 3″. Find the length of the trefoil and the area inclosed by the trefoil.



Ex. 967. The quatrefoil here shown is based upon a square whose side is 5″. Find the length of the quatrefoil and the area inclosed by the quatrefoil.



Ex. 968. Find the circumference of a circle whose area is k sq. ft.

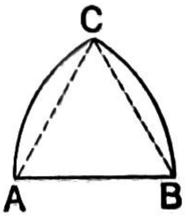
Ex. 969. Find the radius of a circle whose area is equivalent to the sum of the areas of two circles with radii 10″ and 24″ respectively.

Ex. 970. The chord of a segment of a circle is 4″ from the center and is $8\sqrt{3}''$ long. Find the area of the segment.

Ex. 971. Two tangents to a circle form an angle of 80°. If the radius is 10″, find the length of the major arc between the points of contact.

Ex. 972. The area of a certain sector of a circle is equivalent to the square constructed on the radius. What is the angle of the sector?

Ex. 973. *ABC* is an equilateral Gothic arch whose span *AB* is 5 ft. (*A* is the center of $\widehat{BC}$, *B* is the center of $\widehat{AC}$.) Find the area of the arch.



Ex. 974. With the vertices of an equilateral triangle as centers and a side of the triangle as radius, three equal circles are drawn. If a side of the

triangle is k'', find the area of the figure which is common to the three circles.

Ex. 975. A side of an equilateral triangle is $7''$. Find the area of the inscribed circle; of the circumscribed circle.

Ex. 976. A circle of $5''$ radius has the same perimeter as an equilateral triangle. Find the ratio of their areas.

Ex. 977. A circle of $5''$ radius has the same area as an equilateral triangle. Find the ratio of their perimeters.

Ex. 978. Three equal circles are drawn, each tangent to the other two. If the common radius is r, what is the area contained between the three circles?

Ex. 979. In a circle whose radius is $2''$ find the area of a segment whose arc is 120°.

Ex. 980. The circumferences of two concentric circles are 26π and 30π respectively. Find the area of the ring included between the circles.

Ex. 981. The area of the ring between two concentric circles is 5π sq. ft., and the distance between the circles is 1 ft. Find the radii of the circles.

Ex. 982. A chord of the larger of two concentric circles intercepts one third of the circumference and is tangent to the smaller circle. Find the ratio of the areas of the two circles.

Ex. 983. If s is a side of a regular polygon inscribed in a circle of radius r, find the apothem.

Ex. 984. Given a circle of radius r, and s the side of a regular inscribed polygon. Find the side of the regular circumscribed polygon of the same number of sides.

Ans. $\dfrac{2rs}{\sqrt{4r^2 - s^2}}$.

Ex. 985. Given a circle of radius r, find the length of a side of the regular inscribed triangle; of the regular circumscribed triangle. *Ans.* $r\sqrt{3}$; $2r\sqrt{3}$.

Ex. 986. Find the side of a regular hexagon circumscribed about a circle whose radius is r. *Ans.* $\frac{2}{3}r\sqrt{3}$.

Ex. 987. A side of a regular octagon is 6″. Find the radius of the octagon.

Ex. 988. Find the area of the circle inscribed in an equilateral triangle whose side is b.

Ex. 989. Find the number of square feet of tin that would be wasted in cutting the largest possible circular disc from a piece in the form of an equilateral triangle 8″ on a side.

Ex. 990. Find the area of the circle inscribed in a rhombus whose perimeter is 100″ and longer diagonal 40″.

Ex. 991. Two tangents to a circle from an external point form an angle of 60°, and the chord of contact is 10″. Find the area of the segment formed by the minor arc and the chord of contact.

Ex. 992. With the vertices of a square as centers, four equal circles are drawn so that each circle touches two others. The part of the square not covered by the circles equals 85.84 sq. in. Find the radius of each circle.

Ex. 993. The sum of a side of the regular triangle circumscribed about a circle and of a side of the regular inscribed triangle is 12″. Find the radius of the circle.

Ex. 994. Given a circle of radius r, find the length of a side of the regular inscribed octagon; of the regular circumscribed octagon. *Ans.* $r\sqrt{2-\sqrt{2}}$; $2r(\sqrt{2}-1)$.

Ex. 995. Given a circle of radius r, find the length of a side of the regular inscribed polygon of 12 sides; of the regular circumscribed polygon of 12 sides. *Ans.* $r\sqrt{2-\sqrt{3}}$; $2r(2-\sqrt{3})$.

Ex. 996. The distance between two parallel sides of a regular hexagon is d. Find the area of the hexagon.

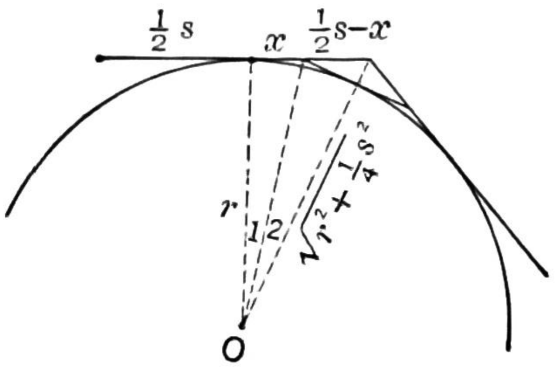
Ex. 997. Find the area included between two parallel chords of a circle whose radius is 5″, if one chord is a side of an inscribed square and the other is a side of a regular inscribed triangle. (Two solutions.)

Ex. 998. Given a circle of radius r, and that the side of an inscribed regular n-gon is s. Find the length of a side of the regular inscribed $2n$-gon. *Ans.* $\sqrt{2r^2 - r\sqrt{4r^2 - s^2}}$.

Ex. 999. Given a circle of radius r, and that the side of a circumscribed regular n-gon is s. Find the length of a side of the regular circumscribed $2n$-gon.

$$Ans.\ \frac{2rs}{2r + \sqrt{4r^2 + s^2}}$$

Sug. $\angle 1 = \angle 2$
$\therefore$ apply 230.
$2x$ is the required length.



Miscellaneous Constructions

Ex. 1000. Construct a semicircle equivalent to a given circle.

Ex. 1001. Form a regular hexagon from an equilateral triangle by cutting off the corners.

Ex. 1002. Inscribe a circle in a given sector of a circle.

Ex. 1003. In a given circle inscribe three equal circles, each tangent to two others and to the given circle.

Ex. 1004. In a given equilateral triangle inscribe three equal circles, each tangent to the two others and to two sides of the triangle.

Ex. 1005. Construct a circle equivalent to two thirds of a given circle.

Ex. 1006. Construct a circle equivalent to the area bounded by two concentric circles.

Ex. 1007. Through a given point draw a line which shall divide the circumference of a given circle in the ratio 3 : 5.

Ex. 1008. Draw three concentric circles whose circumferences are to each other as 1 : 2 : 3.

Ex. 1009. Draw three concentric circles whose areas are to each other as 1 : 2 : 3.

Ex. 1010. Construct a circle whose area shall be to the area of a given circle in the given ratio $m : n$.

Ex. 1011. Divide a circle into three equivalent parts by means of concentric circles.

Ex. 1012. On a given line as a side, construct a regular polygon of 12 sides.

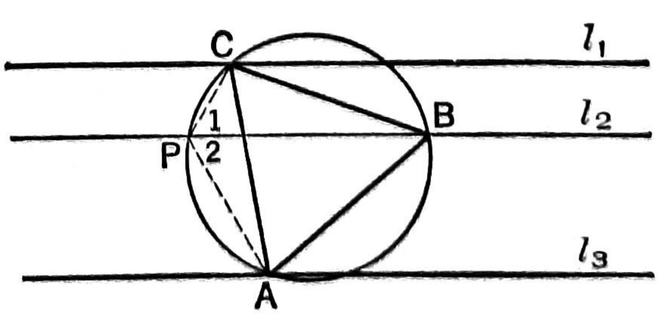
Ex. 1013. Construct a regular hexagon equivalent to one third of a given regular hexagon.

Ex. 1014. Inscribe a square in a sector whose central angle is a right angle.

Ex. 1015. Inscribe a regular octagon in a given square.

Ex. 1016. Construct the locus of the center of a circle, radius $\frac{1}{2}''$, which rolls around an equilateral triangle, altitude $2''$. Compute, correct to one decimal place, the perimeter of the locus and the area inclosed by the locus.

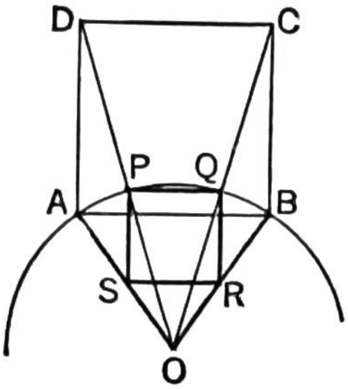
Ex. 1017. Construct an equilateral triangle so that its vertices shall lie in three given parallel lines.



Analysis. Assume the problem done, and let ABC be the required triangle. If we circumscribe a ⊙ about $\triangle ABC$, $\angle 1 = 60°$, $\angle 2 = 60°$ (why?). Suggesting that we take any point P in l_2, construct $\angle 1 = 60°$, and $\angle 2 = 60$, circumscribe a ⊙ about $\triangle PAC$, etc.

Ex. 1018. Inscribe a square in a given sector of a circle, so that two of its vertices shall lie on the intercepted arc.

Solution. We shall make use of the properties of similar figures. With AB as a side, construct square $ABCD$. Draw OC and OD, cutting the circle at Q and P respectively. Draw PQ. Erect a perpendicular to PQ at P, and let it meet OA at S. Erect a perpendicular to PQ at Q, and let it meet OB at R. Draw RS. Then $PQRS$ is the required square. For proof, it is necessary to prove $PQ \parallel CD$, and then make use of pairs of similar triangles, etc.



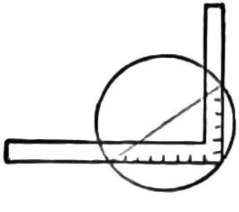
Practical Applications

Ex. 1019. What must be the length of a bar of steel to make a rim for a carriage wheel $4'$ in diameter, allowing $1''$ for weld?

Ex. 1020. How large a pipe is required to carry twice as much water as a $3''$ pipe?

Note: The amount of flow varies as the area of the cross-section of the pipe. A $3''$ pipe is one whose diameter is $3''$.

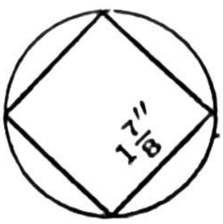
Ex. 1021. It is desired to replace a $2''$ pipe and a $1\frac{1}{2}''$ pipe by a single pipe that will carry as much water as the two pipes together. How large a pipe is needed?



Ex. 1022. Show how the above example may be solved by means of a carpenter's square.

Ex. 1023. Find the total pressure on the bottom of a cylindrical tank $8'\ 6''$ in diameter, if the pressure is 930 lbs. per sq. ft.

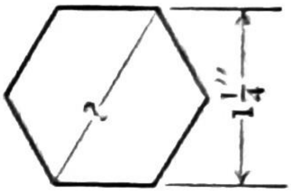
Ex. 1024. A steel rod can safely stand a pull of 16,000 lbs. per square inch. What must be the diameter of a rod to carry 120,000 lbs.?



Ex. 1025. What must be the diameter of round stock so that a square bolt head $1\frac{7}{8}''$ on a side may be milled from it?

Ex. 1026. Find the distance across the

flats of the square head of a cap screw that may be milled from round stock $1\frac{1}{4}''$ in diameter.



Ex. 1027. A hexagonal nut is $1\frac{1}{4}''$ across the flats. Find the distance across the corners.

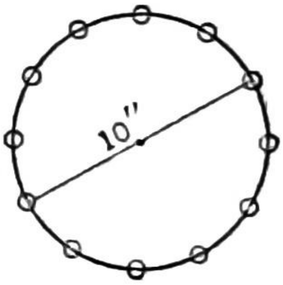
Ex. 1028. Prove that the distance across the corners of a hexagonal nut is approximately 1.15 times the distance across the flats.

Ex. 1029. What is the waste in cutting the largest possible circular plate from a piece of sheet steel 16″ by 18″?

Ex. 1030. Find the diameter of a circular shaft so that it may have a triangular end 3″ on a side.

Ex. 1031. A circular walk 10′ wide has an inside diameter of 120′. How much will it cost to pave the walk at $3.80 per sq. yd.?

Ex. 1032. Find the area of cross-section of a pipe $2\frac{3}{4}''$ in diameter, if the metal is $\frac{3}{8}''$ thick.



Ex. 1033. Twelve bolt holes are to be drilled in a cylinder head on the circumference of a 10″ circle. What is the straight line distance between centers of holes?

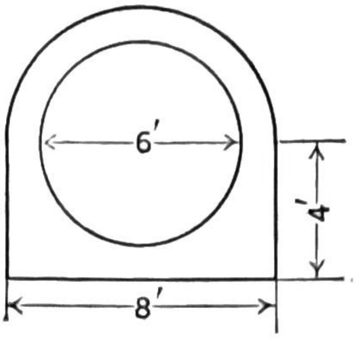
Ex. 1034. How many iron pickets are required to fence a circular plot of ground 84′ in diameter, the distance between adjacent pickets being 6″?

Ex. 1035. A locomotive wheel 5′ in diameter made 10,000 revolutions in a distance of 24 miles. What distance was lost due to the slipping of the wheels?

Ex. 1036. The minute-hand of a clock is twice as long as the hour-hand. Show that the speeds of their extremities are as 24 : 1.

Ex. 1037. The diameter of the earth being 8000 miles, show that the velocity of a body at the equator due to the earth's rotation is about 17.5 miles per minute.

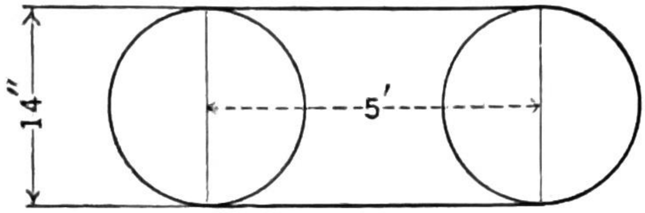
Ex. 1038. A concrete conduit has the cross-section shown in the annexed figure. Find the area of the cross-section.



Ex. 1039. Find the area of the cross-section of the egg-shaped sewer in Ex. 424, if the width AB is 8′.

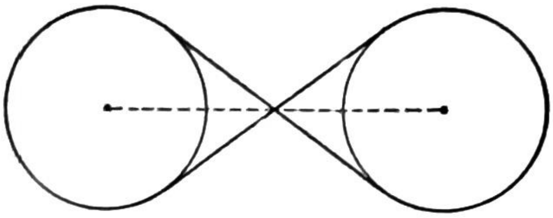
Ex. 1040. Find the length of an open belt connecting two 14″ pulleys 5′ center to center.

Ex. 1041. Find the length of the arc of contact of a belt with a pulley, if the pulley is 18″ in diameter and the arc of contact is 190°.



Ex. 1042. Find the length of an open belt connecting two pulleys 10′ apart, and having diameters of 12′ and 2′ respectively.

Ex. 1043. Find the length of a crossed belt connecting two 8′ pulleys 16′ center to center.

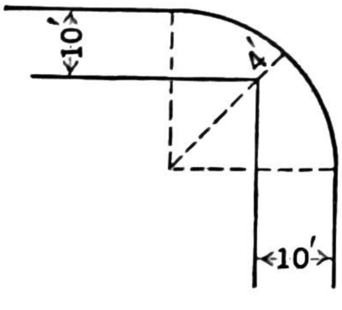


Ex. 1044. Find the length of a crossed belt connecting two pulleys 6′ apart, and having diameters of 4′ and 2′ respectively.

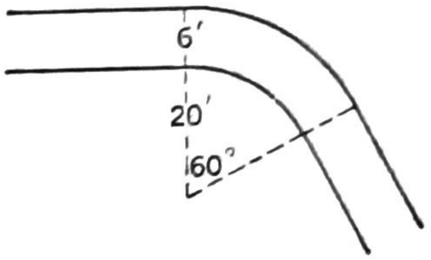
Ex. 1045. A pendulum 21″ long swings through an arc of 40°. How long an arc does its end describe?

Ex. 1046. In laying a curved track, a rail 200 ft. long is bent through an angle of 20°. What is the radius of curvature?

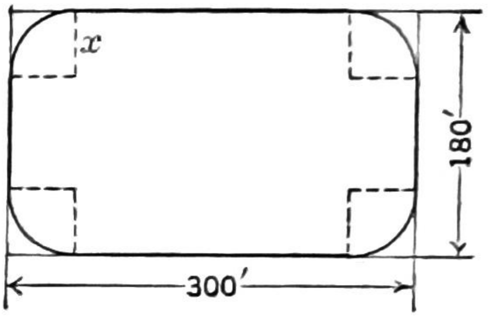
Ex. 1047. The rail of a street-car track is 10′ from the curb. In rounding a square corner it is 4′ from the curb. What is the radius of curvature of the curve?



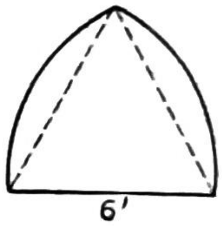
Ex. 1048. A sidewalk 6′ wide turns a corner. Find the area of the curved portion of the sidewalk if the inner radius is 20′ and the central angle is 60°.



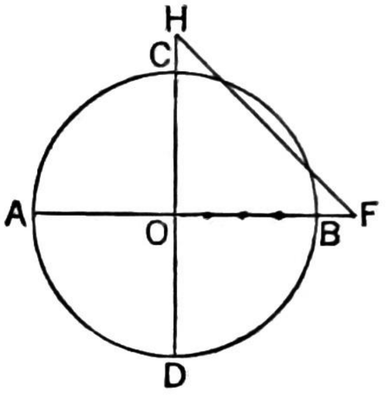
Ex. 1049. It is desired to lay out a running track which will take six laps for a mile. If the dimensions are as given in the annexed figure, and if the corners are to be rounded off by equal quadrants, how large a radius must be used? How many square feet are inclosed by the track?



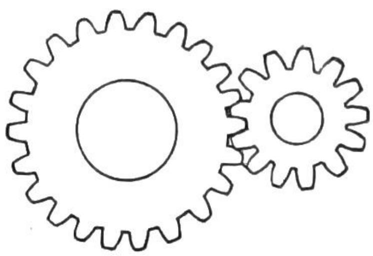
Ex. 1050. How many square feet of glass would be required for a window that is in the shape of an equilateral Gothic arch, if the span is 6 feet?



Ex. 1051. To find the side of a square equivalent to a given circle, sheet metal workers sometimes use the following rule: Draw diameter $AB \perp$ diameter CD. Divide OB into four equal parts. Prolong OB to F so that BF equals one of these parts. Prolong OC to H so that CH equals one of these parts. Draw FH. Then FH is the side of the required square. Show that this method gives a result that is about .53% too small.



Ex. 1052. $ABCDEF$ is a regular hexagon. Show that the resultant of the system of forces represented by AB, AD, AE is 2 AD. (See Ex. 200.)



Ex. 1053. When one gear drives another, the number of revolutions per minute (r.p.m.) varies inversely as the number of teeth. In other words, the smaller the gear the greater the number of revolutions per minute. The adjoining figure shows a pair of gears. Two gears in mesh have 24 teeth and 36 teeth respectively. If the former makes 120 r.p.m., how many r.p.m. does the latter make?

Ex. 1054. A 48-tooth gear running 320 r.p.m. is required to drive another gear at 240 r.p.m. How many teeth must the driven gear have?

Ex. 1055. When two pulleys are connected by a belt, the size of the pulleys varies inversely as their r.p.m. The smaller of two belted pulleys is 8″ in diameter and makes 150 r.p.m. How many r.p.m. does the larger pulley make, if it is 14″ in diameter?

Ex. 1056. The larger of two belted pulleys is 16″ in diameter, and makes 80 r.p.m. What must be the diameter of the smaller pulley to make 120 r.p.m.?

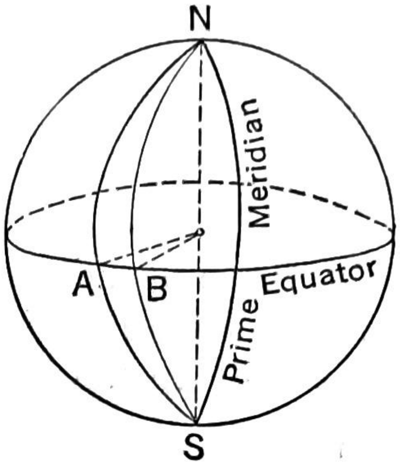
Ex. 1057. The surface speed or rim speed of a wheel is the number of feet that a point on the rim or circumference of the wheel travels in one minute. Obviously during one revolution a point on the rim travels a distance equal to the circumference of the wheel. Hence the number of feet that a point on the rim will travel in one minute must equal the number of revolutions the wheel makes in one minute multiplied by its circumference in feet. Find the rim speed of an emery wheel 14″ in diameter making 180 r.p.m.

Ex. 1058. A flywheel making 330 r.p.m. has a rim speed of a mile a minute. What is its diameter?

Ex. 1059. At how many r.p.m. could a 4-foot grindstone be run if it can stand a surface speed of 700 feet per minute?

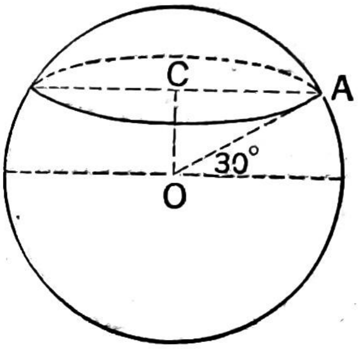
Ex. 1060. Assuming the earth to be a sphere of 8000 miles diameter, how many miles are there between two places on the equator if one is in 55° west longitude and the other is in 61° west longitude?

Sug. The required distance is the length of arc *AB*, which is a 6° arc of the great circle of the equator.



Ex. 1061. How many miles are there in an arc of 1° on the parallel of 30° north latitude?

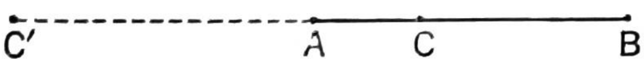
Sug. First find *CA*, the radius of the circle.



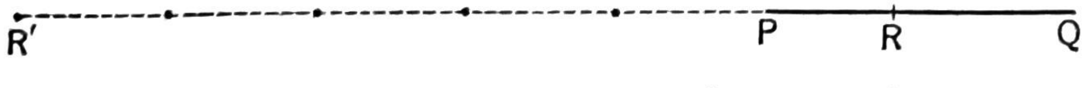
APPENDIX

In this appendix are presented certain propositions and topics not required in the average course in plane geometry. They will appeal to the better student, and at the same time meet the requirements of more exacting courses in geometry.

342. The line AB is divided *internally* at C, if the point C is between A and B. AB is then equal to the sum of its internal segments CA and CB. The line AB is divided *externally* at C', if the point C' is in the prolongation of AB. AB is then equal to the difference of its external segments $C'B$ and $C'A$. Note that in each case the segments are the distances from the point of division to the extremities of the line.



343. If a line is divided internally and externally in the same ratio it is said to be divided *harmonically*. Thus, if $\frac{CA}{CB} = \frac{C'A}{C'B}$ then AB is divided harmonically at C and C'. Suppose it is required to divide a given line PQ harmonically in the ratio 5 : 7. We proceed as follows:



Divide PQ into $5 + 7$, or 12, equal parts. Denote the end of the fifth part by R. Then $\frac{RP}{RQ} = \frac{5}{7}$. Now divide PQ into $7 - 5$, or 2, equal parts, and lay off on the prolongation of PQ, to the left of P, five of these equal parts and denote the end of the fifth part by R'. Then $\frac{R'P}{R'Q} = \frac{5}{7}$.

$\therefore \dfrac{RP}{RQ} = \dfrac{R'P}{R'Q}$. Hence PQ is divided harmonically at R and R' in the required ratio.

Ex. 1062. Divide a given line harmonically in the ratio 3 : 4.

Ex. 1063. Divide a given line harmonically in the ratio $m : n$ (where m and n are two given lines).

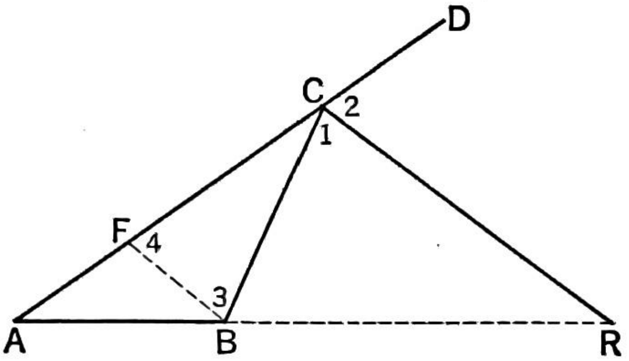
Ex. 1064. If AB is divided harmonically at C and D, prove that CD is divided harmonically at A and B.

Ex. 1065. The common external tangent of two unequal circles divides the centers-segment externally into segments that are to each other as the radii.

BISECTOR OF EXTERIOR ANGLE

Proposition I. Theorem

344. The bisector of an exterior angle of a triangle divides the opposite side externally into segments that are proportional to the other two sides.



Given $\triangle ABC$, with CR the bisector of exterior angle DCB meeting AB produced at R.

Prove $\dfrac{RA}{RB} = \dfrac{CA}{CB}$.

Analysis. The proportion suggests that we draw a line ∥ one side of △ ACR cutting the other two sides. This will enable us to find the ratio RA : RB.

Proof. Draw $BF \parallel CR$, meeting AC at F. Then

(1) $\frac{RA}{RB} = \frac{CA}{CF}$	(1) 224.
(2) $\angle 1 = \angle 3$	(2) 56.
(3) $\angle 2 = \angle 4$	(3) 58.
(4) but $\angle 1 = \angle 2$	(4) Given.
(5) $\therefore \angle 3 = \angle 4$	(5) Ax. 1.
(6) $\therefore CF = CB$	(6) 73.
(7) $\therefore \frac{RA}{RB} = \frac{CA}{CB}$	(7) Sub.

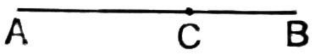
Q.E.D.

Is it possible for CR not to meet AB?

Ex. 1066. The bisectors of the interior and exterior angles at the vertex of a triangle divide the opposite side harmonically.

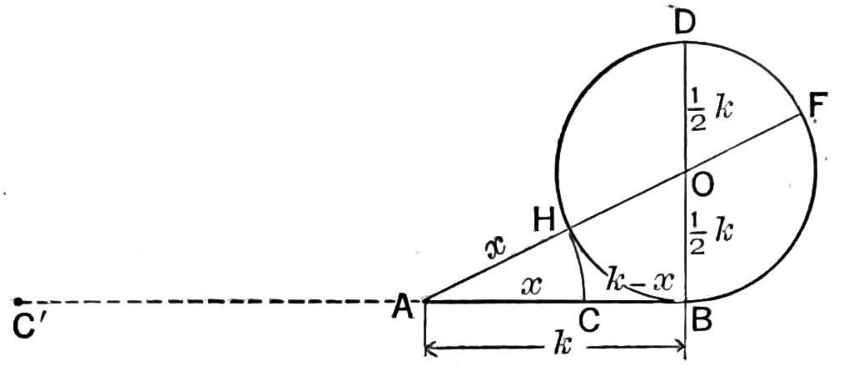
MEAN AND EXTREME RATIO

345. If a point divides a line into two segments such that the greater segment is the mean proportional between the whole line and the smaller segment, the line is said to be divided internally in *mean and extreme ratio.* Thus if C is so situated on AB that $AB : AC = AC : CB$, then AB is divided into mean and extreme ratio. If we denote AB by k, AC by x, CB by $k - x$, we have $k : x = x : k - x$. $\therefore x^2 + kx - k^2 = 0$. Solving this equation we obtain $x = 0.618\,k$, $k - x = 0.382\,k$. Thus the segments are approximately in the ratio 3 : 2. This division is often called the *Golden Section,* and it is claimed by artists that it is especially pleasing to the eye.



Proposition II. Problem

346. To divide a given line in mean and extreme ratio.



Given line AB, its length denoted by k.

Required to find a point C in AB such that $AB : AC = AC : CB$.

Analysis. Assume the problem done, and denote AC *by* x, CB *by* $k - x$. *Then* $k : x = x : k - x$. $\therefore x^2 + kx = k^2$. $\therefore x + k : k = k : x$, *suggesting that* k *be made a tangent to a* ⊙ *of diameter* k, $x + k$ *will be the secant, and* x *the external segment of the secant* (*see* 250).

Construction. (*a*) At B draw $BD \perp AB$, and make $BD = k$.
(*b*) With BD as diameter draw a ⊙.
(*c*) Draw the secant AF through O, the center of the ⊙.
(*d*) On AB lay off $AC = AH$. Then C is the required point.

Proof.

(1) $AF : AB = AB : AH$	(1) 250.
(2) $\therefore AC + AB : AB = AB : AC$	(2) Sub.
(3) $\therefore AC + AB - AB : AB = AB - AC : AC$	(3) 218.
(4) $\therefore AC : AB = CB : AC$	(4) Sub.
(5) $\therefore AB : AC = AC : CB$	(5) 216.
	Q.E.D.

347. If BA be prolonged to C' so that $AC' = AF$, then

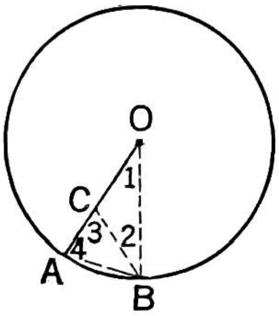
$AF : AB = AB : AH$	250.
$\therefore AF + AB : AF = AB + AH : AB$	217.
$\therefore C'B : C'A = C'A : AB$	Sub.

Hence AB is divided externally at C' in mean and extreme ratio.

REGULAR DECAGON

Proposition III. Problem

348. To inscribe a regular decagon in a given circle.



Given ⊙ O.

Required to inscribe a regular decagon in ⊙ O.

Plan is to divide a radius internally in mean and extreme ratio, and to use the larger segment as a side of the required polygon.

Construction. (*a*) Divide radius OA in mean and extreme ratio by the method of 346, and let CO be the greater segment.

(*b*) With A as center and CO as radius draw an arc cutting the ⊙ at B. Then chord AB is a side of the regular inscribed decagon.

Proof. Draw BO and BC.

(1) $AO : CO = CO : CA$	(1) Const.
(2) $\therefore AO : AB = AB : CA$	(2) Sub.

(3) $\therefore \triangle AOB \sim \triangle ABC$	(3) 244.
(4) $\therefore \triangle ABC$ is isosceles i.e. $AB = BC$	(4) $\triangle AOB$ is isosceles, and $\triangle ABC \sim \triangle AOB$.
(5) but $AB = CO$	(5) Const.
(6) $\therefore BC = CO$	(6) Ax. 1.
(7) $\therefore \angle 1 = \angle 2$	(7) 42.
(8) $\therefore \angle 3 = 2 \angle 1$	(8) 67.
(9) but $\angle 3 = \angle 4$	(9) 42.
(10) $\therefore \angle 4 = 2 \angle 1$	(10) Ax. 1.
(11) $\therefore \angle ABO = 2 \angle 1$	(11) $\angle ABO = \angle 4$ by 42.
(12) but $\angle 1 + \angle 4 + \angle ABO = 180°$	(12) 66.
(13) $\therefore \angle 1 + 2 \angle 1 + 2 \angle 1 = 180°$	(13) Sub.
(14) $\therefore \angle 1 = 36°$ or $\frac{1}{10}$ of 360°	(14) Simplifying.
(15) $\therefore \widehat{AB} = \frac{1}{10}$ of circumference	(15) A central angle is measured by its intercepted arc.
(16) $\therefore$ chord AB is a side of the regular inscribed decagon.	(16) 303.

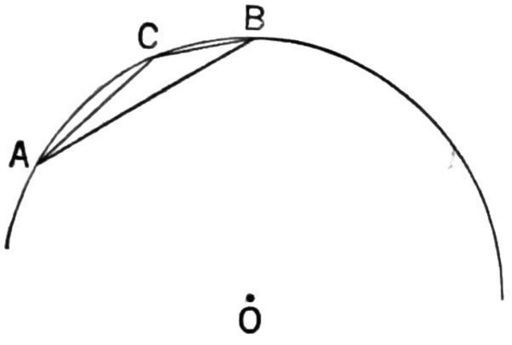
Q.E.D.

349. *Cor.* 1. Starting at any vertex of the regular inscribed decagon, and joining the alternate vertices, we obtain a regular inscribed pentagon.

350. *Cor.* 2. By the method of 304, we obtain regular polygons of 20, 40, 80, etc., sides.

Proposition IV. Problem

351. To inscribe a regular polygon of 15 sides in a given circle.



Given ⊙ O.

Required to inscribe a regular polygon of 15 sides in ⊙ O.

Analysis. The central angle of a regular polygon of 15 *sides is* $\frac{360^\circ}{15}$ *or* 24°, *and* $24^\circ = 60^\circ - 36^\circ$. *This suggests that we combine the constructions of* 308 *and* 348.

Construction. (*a*) Draw AB, a side of the regular inscribed hexagon (308).

(*b*) Draw AC, a side of the regular inscribed decagon (348).

(*c*) Draw BC. Then BC is a side of the required polygon.

Proof.

(1) $\widehat{BC} = \widehat{AB} - \widehat{AC}$	(1) Ax. 6.
(2) $\therefore \widehat{BC} = \frac{1}{6}$ of ⊙ $- \frac{1}{10}$ of ⊙ $= \frac{1}{15}$ of ⊙	(2) Sub.
(3) $\therefore$ chord BC is a side of a regular inscribed polygon of 15 sides.	(3) 303. Q.E.D.

352. *Cor.* By the method of 304, we obtain regular polygons 30, 60, 120, etc., sides.

Ex. 1067. Circumscribe a regular decagon about a given circle.

Ex. 1068. Circumscribe a regular pentagon about a given circle.

Ex. 1069. On a given line as a side, construct a regular decagon.

Ex. 1070. On a given line as a side construct a regular pentagon.

Ex. 1071. Construct an angle of 18°.

Ex. 1072. Two diagonals of a regular pentagon, not drawn from a common vertex, divide each other in mean and extreme ratio.

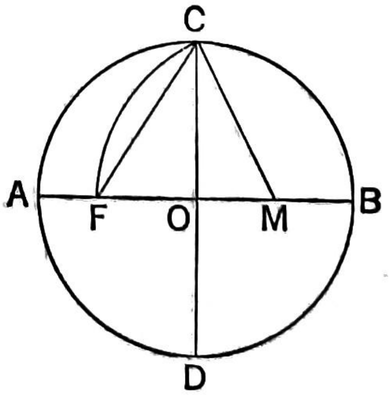
Ex. 1073. Find the length of a side of a regular decagon inscribed in a circle of radius r. *Ans.* $\frac{r}{2}(\sqrt{5}-1)$.

Ex. 1074. Find the apothem of a regular pentagon inscribed in a circle of radius r. *Ans.* $\frac{r}{4}(1+\sqrt{5})$.

Ex. 1075. Find the length of a side of a regular pentagon inscribed in a circle of radius r. *Ans.* $\frac{r}{2}\sqrt{10-2\sqrt{5}}$.

Ex. 1076. Find the area of a regular pentagon inscribed in a circle of radius r. *Ans.* $\frac{5r^2}{8}\sqrt{10+2\sqrt{5}}$.

Ex. 1077. In the annexed drawing, AB and CD are two diameters perpendicular to each other, M is the midpoint of OB, $MF = MC$. Prove that OF is a side of the regular inscribed decagon, and that CF is a side of the regular inscribed pentagon.



Sug. Prove $OF = \frac{r}{2}(\sqrt{5}-1)$, $CF = \frac{r}{2}\sqrt{10-2\sqrt{5}}$.

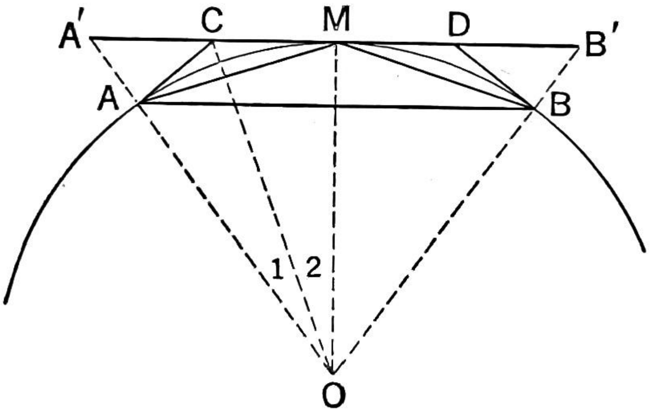
The construction here given is a very convenient one for finding the side of the regular inscribed pentagon and of the regular inscribed decagon.

Ex. 1078. The square of a side of a regular inscribed pentagon is equal to the sum of the squares of the sides of the regular hexagon and of the regular decagon inscribed in the same circle.

PERIMETERS OF 2 n-GONS

Proposition V. Problem

353. To express the perimeters of regular inscribed and circumscribed 2 n-gons in terms of the perimeters of the regular inscribed and circumscribed n-gons.



Solution. Let AB be a side of the regular inscribed n-gon whose perimeter is p_n.

Let $A'B'$ be a side of the regular circumscribed n-gon whose perimeter is P_n.

Let AM be a side of the regular inscribed 2 n-gon whose perimeter is p_{2n}.

Let CD be a side of the regular circumscribed 2 n-gon whose perimeter is P_{2n}.

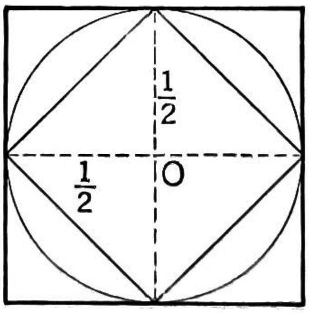
(1) $P_n : p_n = OA' : OM$	(1) 319.
(2) $\angle 1 = \angle 2$	(2) Why?
(3) $\therefore OA' : OM = A'C : CM$	(3) 230.
(4) $\therefore P_n : p_n = A'C : CM$	(4) Ax. 1.

(5) $\therefore P_n + p_n : p_n = A'C + CM : CM = A'M : CM$	(5) 217.
(6) $\therefore P_n + p_n : p_n = \frac{1}{2} A'B' : \frac{1}{2} CD = A'B' : CD = \frac{1}{n} P_n : \frac{1}{2n} P_{2n} = 2P_n : P_{2n}$	(6) Sub.
(7) $P_{2n} = \frac{2P_n \cdot p_n}{P_n + p_n}$. *Ans.*	(7) Solving.
(8) $\triangle$s AMB and ACM are isosceles	(8) Why?
(9) $\therefore \triangle AMB \sim \triangle ACM$	(9) Why?
(10) $\therefore AB : AM = AM : CM$	(10) Why?
(11) $\therefore \overline{AM}^2 = AB \cdot CM$	(11) Why?
(12) $\therefore \left(\frac{1}{2n} p_{2n}\right)^2 = \frac{1}{n} p_n \cdot \frac{1}{4n} P_{2n} = \frac{1}{4n^2} p_n \cdot P_{2n}$	(12) Why?
(13) $\therefore p_{2n} = \sqrt{p_n \cdot P_{2n}}$. *Ans.*	(13) Why?

VALUE OF π

Proposition VI. Problem

354. To compute an approximate value of π.



Solution. Let the diameter of $\odot O$ be 1. Then the perimeter of the inscribed square is $2\sqrt{2}$ or 2.82843, and the perimeter of the circumscribed square is 4.

$$\therefore P_8 = \frac{2P_4 \cdot p_4}{P_4 + p_4} = \frac{8(2.82843)}{4 + 2.82843} = 3.31371 \qquad (353)$$

and $p_8 = \sqrt{p_4 \cdot P_8} = \sqrt{2.82843 \times 3.31371} = 3.06147$ (353)
In like manner we obtain P_{16}, p_{16}; P_{32}, p_{32}; etc. The results of the computation are given in the following table.

Number of sides	*Perimeter of regular circumscribed polygon*	*Perimeter of regular inscribed polygon*
4	4.	2.82843
8	3.31371	3.06147
16	3.18260	3.12145
32	3.15172	3.13655
64	3.14412	3.14033
128	3.14222	3.14128
256	3.14175	3.14151
512	3.14163	3.14157

But the length of the circle is intermediate between the perimeters of the regular inscribed and circumscribed polygons. $\therefore$ the circumference of $\odot O$ is greater than 3.14157 but less than 3.14163. Hence its length, correct to four decimal places, is 3.1416

$$\text{but } \pi = \frac{c}{d} \quad (328)$$

$$\therefore \pi = \frac{3.1416}{1} = 3.1416, \text{ correct to four decimal places.}$$

355. From the earliest times the determination of the value of π has received much attention from mathematicians. Ahmes, an Egyptian priest, about 1700 B.C. gave a value of π equal to 3.1605. The early Hebrews and Babylonians used 3 as the value of π. Archimedes (about 250 B.C.), the greatest mathematician of ancient times, proved that the value of π lies between $3\frac{1}{7}$ and $3\frac{10}{71}$, i.e. between 3.1429 and 3.1408. For most practical work, $3\frac{1}{7}$ is a close enough approximation. In the sixteenth century, Metrus of Holland obtained $\frac{355}{113}$ as the value of π, which is correct to six decimal places. This approximation is easily remembered by writing 113,355 and dividing the last three figures by the first three.

By means of the Calculus mathematicians were enabled more readily to compute the value of π to more and more decimal places, until Shanks in 1873 found its value to 707 decimal places. Such computations go far beyond any practical needs, and are interesting merely from a theoretic point of view. Indeed, ten decimal places for π are sufficient to give the circumference of the earth to the fraction of an inch. The value of π correct to fifteen decimal places is 3.141,592,653,589,793. An exact value of π can never be found, because it has been shown that π is incommensurable (i.e. cannot be expressed as an integer or a fraction) and transcendental (i.e. can not be expressed as a radical or root of an algebraic equation with integral coefficients).

356. *Three famous problems* of geometry have received considerable attention from the time of the early Greeks:

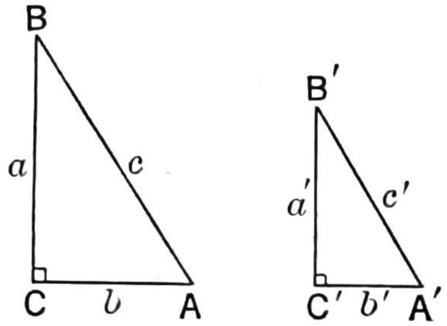
(1) The trisection problem, i.e. to trisect any given angle.

(2) The quadrature problem, i. e. to construct a square equivalent to a given circle.

(3) The duplication problem, i.e. to construct a cube whose volume shall be twice that of a given cube.

All three problems are unsolvable by means of ruler and compasses alone, though all can be solved with the aid of other instruments.

TRIGONOMETRY OF THE RIGHT TRIANGLE



357. Let $\triangle$ ABC and $A'B'C'$ be any two similar right triangles.

Then $\frac{a}{c} = \frac{a'}{c'}$; $\frac{b}{c} = \frac{b'}{c'}$; $\frac{a}{b} = \frac{a'}{b'}$;

i.e. The ratio $\frac{a}{c}$ is the same for all right triangles having the same acute angle A. This constant ratio is called the *sine* of $\angle A$ and is written $\sin A$.

The ratio $\frac{b}{c}$ is the same for all right triangles having the same acute angle A. This constant ratio is called the *cosine* of $\angle A$ and is written $\cos A$.

The ratio $\frac{a}{b}$ is the same for all right triangles having the same acute angle A. This constant ratio is called the *tangent* of $\angle A$ and is written $\tan A$.

The pupil should note that it is customary to denote the right angle by C, hypotenuse by c, the other angles by A and B, and their opposite sides by a and b respectively. We have then in the right triangle ABC

$$\sin A = \frac{a}{c} = \frac{\text{side opposite } \angle A}{\text{hypotenuse}}$$

$$\cos A = \frac{b}{c} = \frac{\text{side adjacent to } \angle A}{\text{hypotenuse}}$$

$$\tan A = \frac{a}{b} = \frac{\text{side opposite } \angle A}{\text{side adjacent to } \angle A}$$

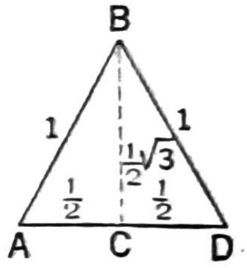
In accordance with the above definitions we have

$$\sin B = \frac{b}{c} = \cos A$$

$$\cos B = \frac{a}{c} = \sin A$$

$$\tan B = \frac{b}{a} = \frac{1}{\tan A}$$

358. It is evident that the values of these ratios will vary with the size of the acute angle, and hence they are called *functions* of the angle. In some cases these functions may be computed by purely geometric methods; e.g. Find sin 60°, cos 60°, tan 60°. Draw any equilateral △ such as ABD, and draw the altitude BC. Then $\angle A = 60°$, and if for convenience we denote a side of the △ by 1, then $AC = \frac{1}{2}$, $BC = \frac{1}{2}\sqrt{3}$. Hence



$$\sin 60° = \frac{CB}{AB} = \frac{\frac{1}{2}\sqrt{3}}{1} = \frac{1}{2}\sqrt{3} = \frac{1}{2}(1.732\ldots.) = 0.866\ldots$$

$$\cos 60° = \frac{AC}{AB} = \frac{\frac{1}{2}}{1} = \frac{1}{2} = 0.5$$

$$\tan 60° = \frac{CB}{AC} = \frac{\frac{1}{2}\sqrt{3}}{\frac{1}{2}} = \sqrt{3} = 1.732\ldots.$$

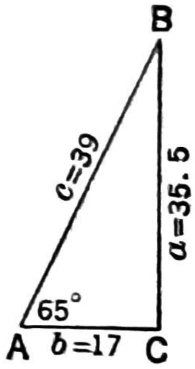
Ex. 1079. Compute sin 30°, cos 30°, tan 30°; sin 45°, cos 45°, tan 45°.

359. The ratios can always be found approximately by measurement; e.g. Find sin 65°, cos 65°, tan 65°. With the aid of a protractor lay off $\angle A = 65°$, and with a scale ruler measure AC, BC, AB. Then

$$\sin 65° = \frac{a}{c} = \frac{35.5}{39} = 0.910 \text{ (table gives 0.906)}$$

$$\cos 65° = \frac{b}{c} = \frac{17}{39} = 0.436 \text{ (table gives 0.423)}$$

$$\tan 65° = \frac{a}{b} = \frac{35.5}{17} = 2.088 \text{ (table gives 2.144)}$$



360. Actually the values in the table given on page 326 are computed by methods of the higher mathematics. This table is correct to four decimal places, and is close enough for most practical purposes.

Four-Place Table of Values of Trigonometric Ratios

Angle	*Sin*	*Cos*	*Tan*	*Angle*	*Sin*	*Cos*	*Tan*
1°	.0175	.9998	.0175	**46°**	.7193	.6947	1.0355
2°	.0349	.9994	.0349	**47°**	.7314	.6820	1.0724
3°	.0523	.9986	.0524	**48°**	.7431	.6691	1.1106
4°	.0698	.9976	.0699	**49°**	.7547	.6561	1.1504
5°	.0872	.9962	.0875	**50°**	.7660	.6428	1.1918
6°	.1045	.9945	.1051	**51°**	.7771	.6293	1.2349
7°	.1219	.9925	.1228	**52°**	.7880	.6157	1.2799
8°	.1392	.9903	.1405	**53°**	.7986	.6018	1.3270
9°	.1564	.9877	.1584	**54°**	.8090	.5878	1.3764
10°	.1736	.9848	.1763	**55°**	.8192	.5736	1.4281
11°	.1908	.9816	.1944	**56°**	.8290	.5592	1.4826
12°	.2079	.9781	.2126	**57°**	.8387	.5446	1.5399
13°	.2250	.9744	.2309	**58°**	.8480	.5299	1.6003
14°	.2419	.9703	.2493	**59°**	.8572	.5150	1.6643
15°	.2588	.9659	.2679	**60°**	.8660	.5000	1.7321
16°	.2756	.9613	.2867	**61°**	.8746	.4848	1.8040
17°	.2924	.9563	.3057	**62°**	.8829	.4695	1.8807
18°	.3090	.9511	.3249	**63°**	.8910	.4540	1.9626
19°	.3256	.9455	.3443	**64°**	.8988	.4384	2.0503
20°	.3420	.9397	.3640	**65°**	.9063	.4226	2.1445
21°	.3584	.9336	.3839	**66°**	.9135	.4067	2.2460
22°	.3746	.9272	.4040	**67°**	.9205	.3907	2.3559
23°	.3907	.9205	.4245	**68°**	.9272	.3746	2.4751
24°	.4067	.9135	.4452	**69°**	.9336	.3584	2.6051
25°	.4226	.9063	.4663	**70°**	.9397	.3420	2.7475
26°	.4384	.8988	.4877	**71°**	.9455	.3256	2.9042
27°	.4540	.8910	.5095	**72°**	.9511	.3090	3.0777
28°	.4695	.8829	.5317	**73°**	.9563	.2924	3.2709
29°	.4848	.8746	.5543	**74°**	.9613	.2756	3.4874
30°	.5000	.8660	.5774	**75°**	.9659	.2588	3.7321
31°	.5150	.8572	.6009	**76°**	.9703	.2419	4.0108
32°	.5299	.8480	.6249	**77°**	.9744	.2250	4.3315
33°	.5446	.8387	.6494	**78°**	.9781	.2079	4.7046
34°	.5592	.8290	.6745	**79°**	.9816	.1908	5.1446
35°	.5736	.8192	.7002	**80°**	.9848	.1736	5.6713
36°	.5878	.8090	.7265	**81°**	.9877	.1564	6.3138
37°	.6018	.7986	.7536	**82°**	.9903	.1392	7.1154
38°	.6157	.7880	.7813	**83°**	.9925	.1219	8.1443
39°	.6293	.7771	.8098	**84°**	.9945	.1045	9.5144
40°	.6428	.7660	.8391	**85°**	.9962	.0872	11.4300
41°	.6561	.7547	.8693	**86°**	.9976	.0698	14.3010
42°	.6691	.7431	.9004	**87°**	.9986	.0523	19.0810
43°	.6820	.7314	.9325	**88°**	.9994	.0349	28.6360
44°	.6947	.7193	.9657	**89°**	.9998	.0175	57.2900
45°	.7071	.7071	1.0000				

HOW TO USE THE TABLE OF TRIGONOMETRIC FUNCTIONS

361. The table gives the values of the sine, cosine, and tangent of angles from 1° to 89° inclusive.

To find sin 24°, for example, look for 24° in the column headed *angle*, and on the same line with 24 in the column headed *sin* is found .4067. Hence sin 24° = 0.4067. The cosine of 24° is opposite 24° in the column headed *cos*. Hence cos 24° = 0.9135.

The tangent of 24° is opposite 24° in the column headed *tan*. Hence tan 24° = 0.4452.

A study of the table shows that the sine and tangent increase as the angle increases, but the cosine decreases as the angle increases.

To obtain the values of functions of angles not in the table we must interpolate. E.g. Find sin 41° 20′

$$\sin 41° = 0.6561$$
$$\sin 42° = 0.6691$$

$\therefore$ sin 41° 20′ is somewhere between 0.6561 and 0.6691. Since 1° produces an increase of 0.6691 − 0.6561 or .0130 (called tabular difference), 20′ or $\frac{1}{3}$ of a degree will produce an increase of $\frac{1}{3}$ of .0130 or .0043. Hence sin 41° 20′ = 0.6561 + .0043 = 0.6604

Find cos 41° 20′.

$$\cos 41° = 0.7547$$
$$\cos 42° = 0.7431$$

$\therefore$ cos 41° 20′ is somewhere between 0.7431 and 0.7547. Since 1° produces a decrease of 0.7547 − 0.7431 or .0116, 20′ will produce a decrease of $\frac{1}{3}$ of .0116 or .0039. Hence cos 41° 20′ = 0.7547 − .0039 = 0.7508.

Find tan 41° 20′.

$$\tan 41° = 0.8693$$
$$\tan 42° = 0.9004$$

$\therefore \tan 41^\circ 20'$ is somewhere between 0.8693 and 0.9004. Since 1° produces an increase of $0.9004 - 0.8693$ or .0311, 20′ will produce an increase of $\frac{1}{3}$ of .0311 or .0104. Hence $\tan 41^\circ 20' = 0.8693 + .0104 = 0.8797$.

Ex. 1080. Find the following:

(1) $\sin 12^\circ$
(2) $\cos 57^\circ$
(3) $\tan 39^\circ$
(4) $\sin 82^\circ 45'$
(5) $\cos 15^\circ 42'$
(6) $\tan 73^\circ 10'$
(7) $\sin 25^\circ 25'$
(8) $\cos 28^\circ 30'$
(9) $\tan 11^\circ 40'$
(10) $\sin 77^\circ 15'$
(11) $\cos 5^\circ 12'$
(12) $\cos 87^\circ 50'$
(13) $\sin 72^\circ 48'$
(14) $\tan 44^\circ 55'$
(15) $\cos 36^\circ 36'$
(16) $\tan 29^\circ 35'$

362. If any of the trigonometric functions are known, then the angle can be found from the tables; e.g. If $\sin x = 0.6397$, find angle x. Looking in the column headed *sin* we find that the nearest smaller number to .6397 is .6293.

$$\sin 39^\circ = 0.6293$$
$$\sin 40^\circ = 0.6428$$

Hence angle x is somewhere between 39° and 40°. The tabular difference is $0.6428 - 0.6293$ or .0135. The difference between $\sin x^\circ$ and $\sin 39^\circ$ is $0.6397 - 0.6293$ or .0104. Since an addition of .0135 increases the angle 1°, an addition of .0104 will increase the angle only $\frac{.0104}{.0135}$ of 1° or $\frac{104}{135}$ of 60′ or 46′ approx. Hence $x = 39^\circ 46'$.

If $\cos x = 0.5610$, find angle x. Looking in the column headed *cos* we find that the nearest smaller number to .5610 is .5592.

$$\cos 56^\circ = 0.5592$$
$$\cos 55^\circ = 0.5736$$

Hence angle x is somewhere between 55° and 56°. The tabular difference is $0.5736 - 0.5592$ or .0144. The difference between $\cos x^\circ$ and $\cos 56^\circ$ is $0.5610 - 0.5592$ or .0018. Since an addition of .0144 decreases the angle 1°, an additio of .0018 will decrease the angle only $\frac{.0018}{.0144}$ of 1° or $\frac{18}{144}$ of 60′ or $7\frac{1}{2}'$. Hence $x = 56° - 7\frac{1}{2}' = 55°\ 52\frac{1}{2}'$.

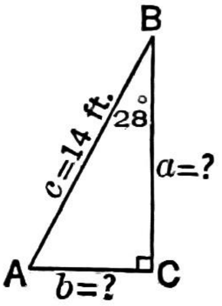
Ex. 1081. Find x in the following equations:

(1) $\sin x = 0.3800$
(2) $\sin x = 0.9762$
(3) $\cos x = 0.2355$
(4) $\cos x = 0.7208$
(5) $\tan x = 0.3350$
(6) $\tan x = 3.9500$
(7) $\sin x = 0.9100$
(8) $\cos x = 0.9100$
(9) $\tan x = 0.9100$
(10) $\cos x = 0.1857$
(11) $\sin x = 0.9887$
(12) $\tan x = 5.8419$

363. The trigonometric ratios are used to find the sides and angles of right triangles.

Ex. 1. In right triangle ABC, $\angle B = 28°$, $c = 14$ ft. Find the remaining parts of the triangle.

The plan is to use that ratio which enables us to express the unknown part in terms of known parts.



Solution.

$$\sin B = \frac{b}{c}$$

$$\begin{aligned} \therefore b &= c \sin B \\ &= 14 \times \sin 28° \\ &= 14 \times 0.4695 \\ &= 6.5730 \text{ ft.} \end{aligned}$$

$$\cos B = \frac{a}{c}$$

$$\begin{aligned} \therefore a &= c \cos B \\ &= 14 \times \cos 28° \\ &= 14 \times 0.8829 \\ &= 12.3606 \text{ ft.} \end{aligned}$$

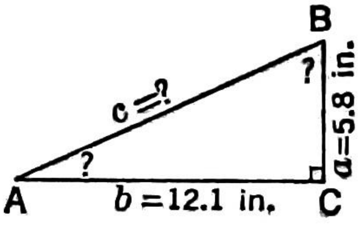
Ans. $A = 62°$, $a = 12.3606$ ft., $b = 6.5730$ ft.

Ex. 2. In right triangle ABC, $a = 5.8$ in., $b = 12.1$ in. Find the remaining parts.

Solution.

$$\tan A = \frac{a}{b} = \frac{5.8}{12.1} = 0.4793$$

$\therefore A = 25°\ 36'$ (to the nearest minute)



$$\tan B = \frac{b}{a} = \frac{12.1}{5.8} = 2.0862$$

$$\therefore B = 64° 23' \text{ (to the nearest minute)}$$

The pupil should note that the sum of A and B must be 90°. The computation actually gives $A + B = 25° 36' + 64° 23' = 89° 59'$. The error of $1'$ arises from our rounding off the angles to the nearest minute, and from the degree of approximation of a four-place table.

c may be found from the equation $c = \sqrt{a^2 + b^2}$ or as follows:

$$\sin A = \frac{a}{c}$$

$$\therefore c = \frac{a}{\sin A} = \frac{5.8}{\sin 25° 36'} = \frac{5.8}{0.4321} = 13.4 \text{ (to the nearest tenth)}$$

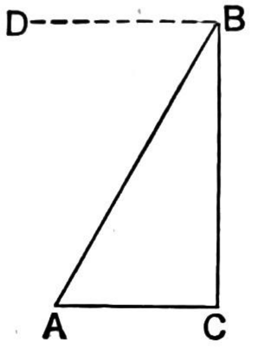
Ans. $A = 25° 36'$, $B = 64° 24'$, $c = 13.4$ in.

Ex. 1082. Solve each of the following right triangles:

(1) Given $A = 29°$, $c = 32$ ft. Find B, a, b.
(2) Given $A = 71°$, $a = 11.3$ in. Find B, b, c.
(3) Given $A = 34° 15'$, $b = 98$ ft. Find B, a, c.
(4) Given $B = 62° 30'$, $c = 18.6$ in. Find A, a, b.
(5) Given $B = 14° 40'$, $b = 57$ yds. Find A, a, c.
(6) Given $B = 52°$, $a = 9.4$ ft. Find A, b, c.
(7) Given $a = 75$ ft., $c = 116$ ft. Find A, B, b.
(8) Given $b = 11$ in., $c = 18$ in. Find A, B, a.
(9) Given $a = 3.4$ ft., $b = 8.5$ ft. Find A, B, c.

364. In actual field practice the surveyor uses a tape or chain to measure distances, and a transit to measure angles. From the measurements thus obtained, he can compute various heights and distances by solving right triangles.

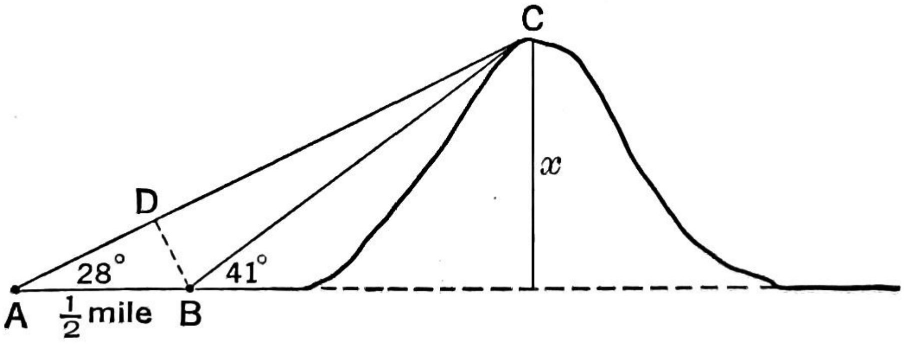
365. Angles of elevation and depression are used to find heights of objects. If CB represents a tower on level ground, and if the observer is at A, $\angle CAB$ is called the *angle of elevation* of point B at point A. If the observer is at B, and if DB is an imaginary line parallel to AC, $\angle DBA$ is called the *angle of depression* of point A at point B. Obviously the angle of elevation must equal the angle of depression.



Ex. 1083. A tower stands on level ground. At a point 200 ft. from the base of the tower the angle of elevation of the top of the tower is 67°. What is the height of the tower?

Ex. 1084. From a lighthouse 275 ft. high, the angle of depression of a ship is 22°. How far is the ship from a point immediately below the lighthouse?

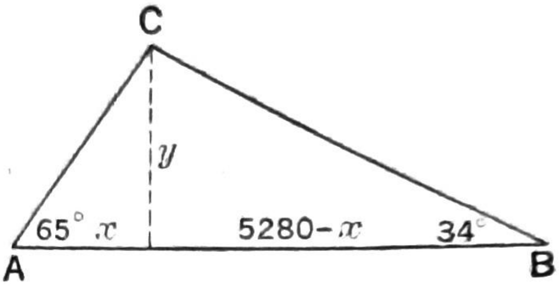
Ex. 1085. A tree 54 ft. high casts a shadow 20 ft. long. What is the angle of elevation of the sun?



Ex. 1086. *A* and *B* are two points half a mile apart on a stretch of level road that leads up to a steep mountain. The angle of elevation of the top of the mountain (C) is 28° at A and 41° at B. How high is the mountain above the level road?

Sug: Draw $BD \perp AC$, find BD, BC, and x.

Ex. 1087. A and B are two stations one mile apart, and at the same moment the angle of elevation of a balloon (C) is 65° at A and 34° at B. How high is the balloon?



Solution.

$$y = x \tan 65^\circ$$

$$y = (5280 - x) \tan 34^\circ$$

$$\therefore x \tan 65^\circ = (5280 - x) \tan 34^\circ$$

$$\therefore x = \frac{5280 \tan 34^\circ}{\tan 65^\circ + \tan 34^\circ}$$

$$\therefore y = \frac{5280 \tan 34^\circ \tan 65^\circ}{\tan 65^\circ + \tan 34^\circ}$$

Let the pupil finish the computation.

366. A great variety of geometric problems can be solved by use of the right triangle.

Ex. 1088. Two sides of a triangle are 16 ft. and 34 ft. respectively, and the angle included between them is 33°. Find the area of the triangle.

Ex. 1089. Find the area of a parallelogram if two adjacent sides, 5″ and 18″ respectively, include an angle of 57°.

Ex. 1090. Find the area of an isosceles triangle whose vertex angle is 40° and whose base is 62 ft.

Ex. 1091. If one leg of a right triangle is 5″ and the opposite angle is 18°, find (*a*) the altitude upon the hypotenuse; (*b*) the other two sides; (*c*) the segments formed by the altitude upon the hypotenuse; (*d*) the area of the triangle.

Ex. 1092. In a circle of 5″ radius find the length of a chord which subtends an arc of 36°.

Ex. 1093. The midpoint of a chord 20″ long is 6″ from the midpoint of the subtended arc. How many degrees are there in the arc?

Ex. 1094. In a circle of 8″ radius find the length of a side of a regular inscribed pentagon. Also find the area of the pentagon.

Ex. 1095. Find the side and area of a regular octagon circumscribed about a circle whose radius is 18 ft.

Ex. 1096. Find the area of a segment of a circle, if the chord of the segment is 18″ and the height of the segment is 5″.

Ex. 1097. Find the length of an open belt connecting two pulleys whose diameters are 20″ and 14″ respectively and whose centers are 35″ apart.

Ex. 1098. Compute the above example if the belt is crossed.

IMPORTANT FORMULAS IN PLANE GEOMETRY

Length of Lines

(1) In a right $\triangle$, $c = \sqrt{a^2 + b^2}$ 254
c is the hypotenuse, a and b are the legs.

(2) In an equilateral $\triangle$, $h = \frac{b}{2}\sqrt{3}$ Ex. 499
h is the altitude, b is a side.

(3) In a square, $d = a\sqrt{2}$ 254
d is the diagonal, a is a side.

(4) In any triangle, $h_c = \frac{2}{c}\sqrt{s(s-a)(s-b)(s-c)}$ Ex. 536
h_c is the altitude on c; a, b, c are the sides; $s = \frac{1}{2}(a + b + c)$.

(5) $m_c = \frac{1}{2}\sqrt{2a^2 + 2b^2 - c^2}$ Ex. 538
m_c is the median on c.

(6) $t_C = \frac{2}{a + b}\sqrt{abs(s - c)}$ Ex. 542
t_C is the bisector of $\angle C$.

(7) In a $\odot$, $c = \pi d$, or $c = 2\pi r$ 328
c is the circumference, d the diameter, r the radius.

(8) length of arc $= \frac{\text{central } \angle}{360^\circ}$ of $2\pi r$ 330

(9) In any regular polygon,
$s_{2n} = \sqrt{2r^2 - r\sqrt{4r^2 - s^2}}$
s is a side of regular inscribed n-gon Ex. 998
s_{2n} is a side of regular inscribed $2n$-gon
r is radius of circumscribed $\odot$.

(10) $s_{2n} = \frac{2rs}{2r + \sqrt{4r^2 + s^2}}$ Ex. 999
s is a side of regular circumscribed n-gon
s_{2n} is a side of regular circumscribed $2n$-gon
r is radius of inscribed circle.

Area of Figures

In the following formulas K denotes area of the figure.

(11) Rectangle, also parallelogram, $K = bh$ 272, 279

b is the base, h the altitude.

(12) Square, $K = a^2$ 272

a is a side.

(13) Any triangle, $K = \frac{1}{2}\, bh$ 284

b is the base, h the altitude.

(14) Any triangle,

$K = \sqrt{s(s-a)(s-b)(s-c)}$ Ex. 778

a, b, c are the sides; $s = \frac{1}{2}(a+b+c)$.

(15) Equilateral △, $K = \frac{1}{4}\, a^2\sqrt{3}$ Ex. 691

a is a side.

(16) Trapezoid, $K = \frac{1}{2}\, h(b + b')$ 290

b and b' are the bases; h the altitude.

(17) Regular polygon, $K = \frac{1}{2}\, ap$ 321

a is the apothem, p the perimeter.

(18) Circle, $K = \pi r^2$ or $K = \frac{1}{4}\, \pi d^2$ 333

r is the radius, d the diameter.

(19) Sector of a ⊙, $K = \frac{1}{2}\, r \times$ length of arc. 337

(20) $K = \dfrac{\text{central } \angle}{360^\circ}\, \pi r^2.$ 336

(21) Segment of a ⊙, $K =$ sector $-$ △. 338

TABLE OF SQUARE ROOTS

No.	*Square root*	*No.*	*Square root*
1	1.000	51	7.141
2	1.414	52	7.211
3	1.732	53	7.280
4	2.000	54	7.348
5	2.236	55	7.416
6	2.449	56	7.483
7	2.646	57	7.550
8	2.828	58	7.616
9	3.000	59	7.681
10	3.162	60	7.746
11	3.317	61	7.810
12	3.464	62	7.874
13	3.606	63	7.937
14	3.742	64	8.000
15	3.873	65	8.062
16	4.000	66	8.124
17	4.123	67	8.185
18	4.243	68	8.246
19	4.359	69	8.307
20	4.472	70	8.367
21	4.583	71	8.426
22	4.690	72	8.485
23	4.796	73	8.544
24	4.899	74	8.602
25	5.000	75	8.660
26	5.099	76	8.718
27	5.196	77	8.775
28	5.292	78	8.832
29	5.385	79	8.888
30	5.477	80	8.944
31	5.568	81	9.000
32	5.657	82	9.055
33	5.745	83	9.110
34	5.831	84	9.165
35	5.916	85	9.220
36	6.000	86	9.274
37	6.083	87	9.327
38	6.164	88	9.381
39	6.245	89	9.434
40	6.325	90	9.487
41	6.403	91	9.539
42	6.481	92	9.592
43	6.557	93	9.644
44	6.633	94	9.695
45	6.708	95	9.747
46	6.782	96	9.798
47	6.856	97	9.849
48	6.928	98	9.899
49	7.000	99	9.950
50	7.071	100	10.000

INDEX

(The numbers refer to pages.)

Adjacent angles, 4
Alternation, 176
Altitude, 16, 241
Analysis, 17, 19
  of exercises, 82
Angle, 4
  acute, 5
  at center of circle, 97
  at center of regular polygon, 283
  bisector of, 6, 44, 186, 216
  degree of, 6
  inscribed in circle, 124–127
  inscribed in segment, 124
  obtuse, 5
  right, 5
  sides of, 4
  straight, 5
  vertex of, 4
Angles
  adjacent, 4
  alternate exterior, 28
  alternate interior, 28
  complementary, 5
  corresponding, 28
  designation of, 4
  equal, 6
  exterior of triangle, 26, 313
  exterior, 28
  interior, 28
  measurement of, 123–133
  of a polygon, 68
  supplementary-adjacent, 4
  vertical, 5, 13
Antecedent, 175
Apothem, 283
Arc, 96, 123
  major, 96
  minor, 96
Area of a figure, 241
  of a circle, 293
  of parallelogram, 244
  of polygon, 241
  of rectangle, 242
  of regular polygon, 288
  of trapezoid, 248
  of triangle, 246
Axioms, 2, 3

Base of isosceles triangle, 15
Bisector of angle, 6
Bisector of angles of triangle, 78, 186, 216

Centers-segment, 117
Center of circle, 96
  of regular polygon, 283
Central angles, 97–99
Centroid, 82
Chord, 99
Chord of contact, 117
Circle, 96
  arc of, 96
  area of, 293
  center of, 96
  chords of, 99–105
  circumference, 96
  circumscribed, 106, 281
  determined, 112
  diameter of, 96
  inscribed, 116, 140, 282
  measurement of, 289
  radius of, 96
  secant of, 113
  sector of, 294
  segment of, 124, 294
  tangents to, 114–117
Circles, concentric, 106
  intersecting, 122

Circles—*continued*
  tangent, externally, 118
  tangent, internally, 118
Circum-center, 113
Circumference, 96
  measurement of, 291
Circumscribed circle, 106
Circumscribed polygon, 116
Complementary angles, 5
Concentric circles, 106
Concurrent lines, 78–82
Congruent figures, 15
Consequent, 175
Constructions, 7, 137–141
  fundamental, 51–58
  methods of attack, 145–151
  of triangles, 59–61, 142–145
Converse theorem, 32
Corollary, 22

Decagon, regular, 316–317
Degree, of angle, 6, 123
  of arc, 123
Determined circle, 112
Determined line and point, 1, 2
Diagonal, 16, 45
Diameter, 96
Distance between two points, 1
  from point to line, 42

Equiangular polygon, 274
Equidistant, 42
Equilateral polygon, 274
Equilateral triangle, 14
Equivalent figures, 257
Exterior angle of polygon, 69
  of triangle, 26, 27
Extremes, 175

Formulas of plane geometry, 334–335
Fourth proportional, 175, 181
Fundamental Constructions, 51–58

Geometry, 1
  subject matter of, 1
  use of, 1

Harmonic division, 312–314
Hexagon, 278
Historical notes
  Euclid, 255
  Gauss, 280
  Pythagoras, 202, 254
  value of π, 322
Hypotenuse, 26

Identity, 6
Indirect proof, 30
Inequalities, 71–77
Inscribed angle, 124
Inscribed circle, 116
Inscribed polygon, 106
Instruments, 4, 51
Inversion, 176
Isosceles trapezoid, 85
Isosceles triangle, 15
  base of, 15
  legs of, 15

Length of secant, 113
  of tangent, 116
Line, 1
  bisected, 52
  curved, 96
  determined, 1
  divided externally, 312
  divided harmonically, 312
  divided in extreme and mean ratio, 314
  divided internally, 65, 182, 312
  of centers, 117
  segment, 62
Lines, 1
  concurrent, 78–82
  parallel, 2
  perpendicular, 5
Loci, 154
  intersection of, 159
Locus, finding of a, 154, 155
  of a point, 154, 155
  problem, solution of, 156

Major arc, 96
Mean and extreme ratio, 314–315
Mean proportional, 176
Means, 175
Measurement
  of angle, 6, 123
  of arc, 123
  of circle, 289
  of circumference, 291
  of rectangle, 242
  of surface, 241
Median, of trapezoid, 85
  of triangle, 15
Minor arc, 96

Numerical relations, 212–219

Originals, How to attack, 82

Parallel lines, 2, 29–32, 35, 63, 134
Parallelogram, 45, 46–49
Pentagon, regular, 317–319
Perpendicular, 5
  bisector, 43
Pi, 291, 321
  evaluation of, 321–323
Point, 1
  determined, 2
  of contact, 114
  of tangency, 114
Polygon, 45
  angles of, 68, 69, 70
  circumscribed, 116
  diagonal of, 45
  equiangular, 274
  equilateral, 274
  exterior angles of, 69
  inscribed, 106
  regular, 274–289
Polygons
  equivalent, 242
  mutually equiangular, 188
  mutually equilateral, 188
  perimeters of, 286
  sides proportional, 187
  similar, 187, 207–211, 252–253, 285
Postulates, 1, 2
  parallel line, 2
  straight line, 1
  transference, 2
Practical applications, 89–95, 171–174, 233–240, 271–273, 306–311
Problem, 6, 51
Projection of line, 212
  of point, 212
Proof, 9
  indirect, 30
  need for, 3
Proportion, 175
  by addition, 177
  by alternation, 176
  by inversion, 176
  by subtraction, 177
  extremes of, 175
  means of, 175
  product of means and extremes, 176
  terms of, 175
Proportional, fourth, 175, 181
  mean, 176, 205
  segments, 178–184
Proposition, definition of, 6
Pythagorean Theorem, 202, 254

Quadrilateral, 45

Radius of circle, 96
  of regular polygon, 283
Ratio, 175
  antecedent of, 175
  consequent of, 175
  mean and extreme, 314
Ratios
  series of equal, 177
Rectangle, 85
  area of, 242
Regular polygon, 274
  angle at center of, 283
  apothem of, 283
  area of, 288

Regular polygon—*continued*
  center of, 283
  radius of, 283
Rhombus, 50
Right triangle, 26
  hypotenuse of, 26
  legs of, 26

Secant, 113
  length of, 113
Sector, 294
Segment, of circle, 124, 294
  of line, 62
Segments of transversals, 63
Semicircle, 96
Series of equal ratios, 177
Similar polygons, 187, 207–211, 252–253, 285
Similar sectors, 294
Similar segments, 295
Similarity of triangles, 188, 193, 194, 250
Square, 16, 62, 277
Straight angle, 5
Straightedge, 4
Straight line, 1
  determined, 1
Supplementary-adjacent angles, 4
Symbols, 12

Table of square roots, 336
Table of trigonometric ratios, 326
Tangents, 114–117
  external common, 118–121
  internal common, 118–121
  length of, 116
Theorem, 6
  converse of, 32
Three famous problems, 323
Transformation, 243
Transversal, 28
Trapezoid, 85
  bases of, 85
  isosceles, 85
  legs of, 85
  median of, 85
Triangle, 14
  altitudes of, 16, 80
  base of, 14
  bisectors of angles of, 78, 186
  centroid of, 82
  circum-center of, 113
  construction of, 59–61, 142–145
  corresponding angles and sides, 15
  equiangular, 14
  equilateral, 14
  exterior angle of, 26, 27
  isosceles, 15, 22
  medians of, 15, 81
  right, 26, 200–202
  sum of angles of, 36
Triangles, Similarity of, 188, 193, 194, 250–251
  congruent, 16, 18, 24, 41
Trigonometry of right angle, 324–333

Unequal chords, 108–111

Vertex of angle, 4
Vertical angles, 5, 13